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SOME UNIQUENESS AND EXACT MULTIPLICITY RESULTS FOR A PREDATOR-PREY MODEL

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ABSTRACT. In this paper, we consider positive solutions of a predator-prey model with diffusion and under homogeneous Dirichlet boundary conditions. It turns out that a certain parameter m in this model plays a very important role. A good understanding of the existence, stability and number of positive solutions is gained when m is large. In particular, we obtain various results on the exact number of positive solutions. Our results for large m reveal interesting contrast with that for the well-studied case $m = 0$, i.e., the classical Lotka-Volterra predator-prey model.

1. INTRODUCTION

In this paper, we shall study the following predator-prey model:

$$\begin{cases} \Delta u + u\big(a - u - bv/(1+mu)\big) = 0 \quad \text{in } \ D, \quad u|_{\partial D} = 0, \\ \Delta v + v\big(d - v + cu/(1+mu)\big) = 0 \quad \text{in } \ D, \quad v|_{\partial D} = 0, \end{cases} \tag{1.1}$$

where D is a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D, a, b, c, d and m are constants with a, b, c positive and m non-negative; d may change sign.

If $m = 0$, then (1.1) is reduced to the classical Lotka-Volterra predator-prey model which has received extensive study in the last decade. See, e.g., [1], [5], [10], [11], [20], [22] and [25]. In particular, the existence of positive solutions for this case was completely understood, see Dancer [10], [11]. It has been conjectured that there is at most one positive solution, but this was shown only for the case that the space dimension n is one, see [24]. For space dimension greater than one, this is still an open problem. We also refer to [14], [21] and [23] for some partial results on uniqueness. The stability of the positive solutions was studied in [14], [21], [25] and [28], but the results are still far from complete.

The case when $m > 0$ was first studied by Blat and Brown [2]. In this case, the term $uv/(1 + mu)$ is known as the Holling-Tanner interaction term, and we refer to [2] for more background on this model. In [2], Blat and Brown studied the existence of positive solutions to (1.1) by making use of both local and global bifurcation theories. Their results coincide with those in [11] when $m = 0$, and therefore are optimal in this case. In a recent paper [3], A. Casal, J.C. Eilbeck and J. Lopez-Gomez also studied problem (1.1) with $m > 0$. The non-existence results in [2] were considerably improved in [3]. In particular, they list two sets of conditions; one is sufficient for the existence of positive solutions of (1.1) (they are a neater version of that given in [2]), while the other is necessary (see Theorem

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3.1 in [3]). Note that there is a gap between these two sets of conditions. If m is relatively small, then the sufficient conditions turn out to be necessary as well (see [3], page 426). Hence in this case the existence problem is completely understood. However, if m is not small, the gap between the necessary conditions and the sufficient ones was left open. In fact, it was observed in [3] that these sufficient conditions are not necessary. Some multiplicity results were also obtained in [3]. By numerical calculations on the model, and also by using local bifurcation theory, they obtained some ranges of the parameters where (1.1) has at least two positive solutions. Furthermore, following ideas in [24], they also established some uniqueness results for the case that the space dimension is one.

The purpose of this paper is to better understand the model for $m > 0$ and not small. In particular, we want to know the exact range of the parameters where (1.1) has a positive solution, and to find the exact number of positive solutions if a non-uniqueness phenomenon appears. We find that for large m, this goal can be rather fully achieved. The point is that when m is large, (1.1) can be viewed as perturbations of some simpler limiting problems which can be easily solved or at least well understood. With the help of these auxiliary problems, one can then use regular or singular perturbation techniques to obtain a good understanding of (1.1). This paper is mainly devoted to the large m case, though we will occasionally tackle the case where m is not large.

We will also study the stability of positive solutions of (1.1). When referring to stability, we shall regard positive solutions of (1.1) as steady-state solutions of the corresponding parabolic problem

$$(1.2)\qquad \begin{cases} u_t - \Delta u + u\big(a - u - bv/(1+mu)\big) = 0 & \text{in } \ D \times (0,\infty), \\ \tau v_t - \Delta v + v\big(d - v + cu/(1+mu)\big) = 0 & \text{in } \ D \times (0,\infty), \\ \qquad u|_{\partial D\times(0,\infty)} = v|_{\partial D\times(0,\infty)} = 0, \end{cases}$$

where τ is a positive constant measuring the ratio of the diffusion rates of the two species u and v. Note that both (1.1) and (1.2) are rescaled versions of the model which appears in [2], hence no generality is lost.

In order to present our main results, we need to introduce some notations and basic facts. For $p \in C^\alpha(\bar{D})$, it is well known that the linear eigenvalue problem

$$-\Delta u + p(x)u = \lambda u \ \text{ in } \ D, \ \ u|_{\partial D} = 0$$

has an infinite sequence of eigenvalues which are bounded from below. We denote the i-th eigenvalue by $\lambda_i(p)$. It is known that $\lambda_1(p)$ is a simple eigenvalue and that the corresponding eigenfunction does not change sign in D. When $p \equiv 0$, we will denote $\lambda_i(0)$ simply by λ_i. Moreover, we denote by Φ_1 the eigenfunction corresponding to λ_1 with normalization $\|\Phi_1\|_\infty = 1$ and positive in D. By the L^p theory of elliptic operators, one easily sees, that $\lambda_i(p)$ is well-defined for any $p \in C(\bar{D})$. Using the variation characterization of the eigenvalues, one can show that (see e.g., [2]) $p \to \lambda_1(p)$ is continuous from $C(\bar{D})$ to R^1 and is strictly increasing in the sense that $p_1 \le p_2$ and $p_1 \not\equiv p_2$ implies $\lambda_1(p_1) < \lambda_1(p_2)$.

It is well-known that for any $a > \lambda_1$, the problem

$$-\Delta u = au - u^2 \ \text{ in } \ D, \ \ u|_{\partial D} = 0$$

has a unique positive solution which we denote by θ_a. It is also known that $a \to \theta_a$ is continuous from (λ_1, ∞) to $C^{2,\alpha}(\bar{D})$, and that $\theta_{a_1} < \theta_{a_2}$ in D if $a_1 < a_2$. Some further properties concerning θ_a will be presented in the beginning of section 2.

Now we are able to state the main results of this paper. Our first result deals with the case $d > \lambda_1$, while the second result is about the case $d \le \lambda_1$.

Theorem A. *Let b, c and $d > \lambda_1$ be fixed. Then there exists some large positive constant M, depending only on b,c and d, such that for each $m \ge M$, there is a unique constant $\tilde{a} \in \big(\lambda_1, \lambda_1(b\theta_d)\big)$ satisfying $\tilde{a} \to \lambda_1$ as $m \to \infty$ and such that*

1) *(1.1) has a positive solution if and only if $a \ge \tilde{a}$;*
2) *if $a = \tilde{a}$ or $a \ge \lambda_1(b\theta_d)$, then (1.1) has a unique positive solution. Moreover, the unique solution is asymptotically stable when $a \ge \lambda_1(b\theta_d)$;*
3) *if $a \in \big(\tilde{a}, \lambda_1(b\theta_d)\big)$, then (1.1) has at least two positive solutions. Furthermore, when $\lambda_1(b\theta_d) \le \lambda_2$, (1.1) has exactly two positive solutions, one asymptotically stable and the other unstable.*

Theorem B. *Suppose that b and c are fixed and $bc \le 4$. Then there exists some large positive constant M, depending only on b and c, such that for each $m \ge M$,*

1) *if $d \le \lambda_1 - c/m$, then (1.1) has no positive solution;*
2) *if $d \in (\lambda_1 - c/m, \lambda_1]$, then there is a unique constant a_0 such that (1.1) has no positive solution for $a \le a_0$, and exactly one positive solution for $a > a_0$. Moreover, this unique solution is asymptotically stable. Here a_0 is uniquely determined by*

$$d = \lambda_1\big(-c\theta_{a_0}/(1 + m\theta_{a_0})\big).$$

Remark 1.1. For large m, Theorem A gives the exact parameter range where (1.1) has at least a positive solution. This fills in the gap left open in [3]. Note that the exact existence range in the case $d \le \lambda_1$ was obtained in [2] already. Our contribution in Theorem B is the uniqueness and stability part.

Remark 1.2. The restriction that $bc \le 4$ in Theorem B can not be removed. In fact, we will show that if bc is relatively large, then for any large m, there exist $d \le \lambda_1$ and $a > \lambda_1$ such that (1.1) has at least three non-degenerate positive solutions.

Remark 1.3. Note that the different features of (1.1) between small m and large m are observed not only in the existence range of positive solutions, they also appear in terms of the number of positive solutions. To be more specific, given any $a > \lambda_1$, b and c, if the space dimension $n = 1$, then it is easy to see from the arguments in [24] that for all small non-negative m, (1.1) has at most one positive solution. We conjecture that this is also true for $n \ge 2$. This contrasts sharply with the case that m is large, where Theorem A and Remark 1.2 assure us that, in both the cases $d > \lambda_1$ and $d \le \lambda_1$, multiplicity can occur.(The multiplicity phenomenon for the case $d > \lambda_1$ was first observed in [3].) Moreover, if $n \ge 2$, then one can use the domain variation technique of Dancer as in [17] to show that in 3) of Theorem A, (1.1) can have a large number of positive solutions if the domain D is chosen properly.

Remark 1.4. For large m, our multiplicity result in Theorem A improves the local results in [3]. In fact part 3) in Theorem A can be considerably strengthened; see Remark 3.1, Theorems 3.2 and 3.3 later for details. Our uniqueness results in Theorems A and B do not imply the uniqueness results of [3] (for the case that the space dimension is one), and vice versa.

The proofs of Theorems A and B are rather lengthy. They are based on the observation that when m is large, then (1.1) can be regarded as a regular perturbation of the decoupled problem

$$\begin{cases}\Delta u + u(a-u) = 0 \quad \text{in } D, \quad u|_{\partial D} = 0,\\ \Delta v + v(d-v) = 0 \quad \text{in } D, \quad v|_{\partial D} = 0,\end{cases} \tag{1.3}$$

and a singular perturbation of the single equation problem

$$-\Delta u = u\big(a - b\theta_d/(1+u)\big) \quad \text{in} \quad D, \; u|_{\partial D} = 0. \tag{1.4}$$

By a well-known result, (1.3) has a unique positive solution (θ_a, θ_d) when $a > \lambda_1, d > \lambda_1$, and there is no positive solution otherwise. We will show in section 2 that (1.4) has a positive solution if and only if $a \in \big(\lambda_1, \lambda_1(b\theta_d)\big)$. Moreover, (θ_a, θ_d) is a stable solution of (1.3) while (1.4) has only unstable ones.

In the following, we lay out the main ideas used in the proofs of Theorems A and B, which, we hope, will be of some help for readers in following the lengthy proofs.

We first consider the proof of Theorem A. If we fix b,c, $d > \lambda_1$ and consider a as a parameter in (1.1), then for m large and $a > \lambda_1$ bounded away from both infinity and λ_1, (1.3) induces a positive solution for (1.1) which is stable. When a falls into the range $\big(\lambda_1, \lambda_1(b\theta_d)\big)$, then (1.4) induces at least an unstable positive solution to (1.1). Moreover, for any positive solution (u, v) of (1.1) with m large, either (u, v) is close to a positive solution of (1.3) or mu is close to a positive solution of (1.4). This gives a rough idea how Theorem A is proved. However, difficulties arise when a is near infinity, close to λ_1 or around $\lambda_1(b\theta_d)$. To overcome these difficulties, we use various careful estimates. In particular, for a near λ_1, both (1.3) and (1.4) gradually lose their influences on (1.1). In fact, from the global bifurcation point of view, one finds that as a approaches λ_1, the stable positive solution branch $\{(a, u, v)\}$ of (1.1) which is close to the solution curve $\Gamma_1 = \{(a, \theta_a, \theta_d)\}$ of (1.3) breaks away from Γ_1. A similar thing happens to the unstable positive solution branch of (1.1) which is induced by (1.4). It turns out that the stable and unstable branches meet at $a = \tilde{a}$, and this is proved by combining careful estimates with a local bifurcation argument. It is interesting to notice that the situation here is very similar to that in Du [17] for a completely different system. In fact, a number of the ideas in the proofs here resemble those in Du [17]. But the detailed techniques are totally different.

The proof of Theorem B involves only uniqueness and stability arguments, since the existence in the case $d \le \lambda_1$ was completely understood in [2]. Uniqueness and stability arguments also appear in the proof of Theorem A. The main idea in these arguments is as follows. By using the fixed point index in cones, one can reduce the proof of uniqueness and stability to the proof of the fact that any possible positive solution is non-degenerate and linearly stable. This is a widely used trick and often involves careful estimates and indirect arguments. See, e.g., [14], [16] and [17]. Certainly, different techniques are required for different problems. In fact, to find appropriate techniques for a given problem is often the hard part in these arguments.

The rest of this paper is organized as follows. In section 2, we give some preliminary results which are needed in the later sections. In section 3, we consider the case $d > \lambda_1$ and prove Theorem A there. Finally in section 4 we study the case $d \le \lambda_1$ and establish Theorem B. We shall also present a counterexample as claimed in Remark 1.2.

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2. Some preliminaries

In this section we present some basic results which will be used in this paper. We will need to know the asymptotic behavior of θ_a as $a \to \lambda_1$ and $a \to \infty$ in later sections. The following result can be easily established by using Theorem 1 of [7].

Lemma 2.1. *$\theta_a(x)/(a-\lambda_1) \to \frac{\int_D \Phi_1^2}{\int_D \Phi_1^3}\Phi_1(x)$ uniformly in $\overline{D}$ as $a \to \lambda_1+$, where Φ_1 is the eigenfunction corresponding to λ_1 with $\max_{\overline{D}} \Phi_1 = 1$.*

Next we study the behavior of θ_a as $a \to \infty$, which turns out to be not as easy.

For $x \in D$ with $d(x) = d(x, \partial D) < \epsilon_0$ and $\epsilon_0 > 0$ small, one can represent x by $x = s + tn(s)$. Here $s \in \partial D$, $n(s)$ is the unit inward normal of ∂D at s and t is a small positive number. In fact $s = s(x)$ and $t = t(x)$ can be uniquely determined, and $t(x) = d(x)$ when $d(x)$ is small enough. Set $\phi_a = \theta_a/a$ for $a > \lambda_1$, and let z_0 be the unique solution of the problem

$$-z'' = z(1-z),\ z(0) = 0,\ z(\infty) = 1. \tag{2.1}$$

Then z_0 is strictly increasing in $[0,\infty)$ and $z_0'' \le 0$. In particular, $kz_0(t) \ge z_0(kt)$ for any $k > 1$ and $t \ge 0$. Now define

$$w_a(x) = \begin{cases} z_0\big(\sqrt{a}t(x)\big) & \text{if } d(x) < \epsilon_0; \\ 1 & \text{if } d(x) \ge \epsilon_0. \end{cases}$$

By Lemma 2 in [13], we have

Proposition 2.1. *For $\delta > 0$, there exists $A = A(\delta) > 0$ such that if $a > A$, then*

$$(1-\delta)w_a(x) \le \phi_a(x) \le (1+\delta)w_a(x). \tag{2.2}$$

By Proposition 2.1 we can easily prove the following result for our later use.

Lemma 2.2. *Given any $\epsilon > 0$ and $k > 1$, there exists A_0 large such that if $a \ge A_0$, then*

$$(1+\epsilon)k^{3/2}\theta_a \ge \theta_{ka}. \tag{2.3}$$

In particular, for any $k_0 \in \big(1, 2^{2/3}\big)$, there exists A_1 large such that if $a \ge A_1$, then

$$2\theta_a \ge \theta_{k_0 a}, \quad \text{and hence} \quad \lambda_1(2\theta_a) \ge \lambda_1(\theta_{k_0 a}) = k_0 a. \tag{2.4}$$

Proof. It suffices to prove (2.3). Choose $\delta > 0$ small such that $(1+\delta)/(1-\delta) < 1+\epsilon$, and let $A = A(\delta)$ be defined as in Proposition 2.1. Then for $x \in D$ with $d(x) < \epsilon_0$ and $a > A$, by (2.2) we obtain

$$\begin{aligned}(1+\epsilon)k^{3/2}\theta_a &= (1+\epsilon)k^{3/2}a\phi_a \ge (1+\delta)k^{3/2}aw_a \\ &= (1+\delta)k^{3/2}az_0\big(\sqrt{a}t(x)\big) \ge (1+\delta)kaz_0\big(\sqrt{ka}t(x)\big) \\ &= (1+\delta)kaw_{ka} \ge ka\phi_{ka} = \theta_{ka}.\end{aligned}$$

For $x \in D$ satisfying $d(x) \geq \epsilon_0$, $\phi_a \to 1$ and $\phi_{ka} \to 1$ uniformly. Hence we can find $A' > 0$ large such that if $a > A'$, then

$$(1+\epsilon)k^{1/2}\phi_a \geq \phi_{ka}.$$

Set $A_0 = \max(A, A')$. Then clearly (2.3) holds for all $a \geq A_0$.#

Sometimes we need to extend the definition of θ_a to functions $a(x)$.

Proposition 2.2. *Let $a(x) \in C^\alpha(\overline{D})$ and $\lambda_1(-a) < 0$. Then the problem*

$$\Delta u + \big(a(x) - u\big)u = 0 \text{ in } D, \quad u|_{\partial D} = 0 \tag{2.5}$$

has a unique positive solution which we still denote as θ_a. Moreover, (2.5) has no positive solution if $\lambda_1\big(-a(x)\big) \geq 0$.

The proof of Proposition 2.2 is quite standard. We refer to [6] for details. One very useful consequence of Proposition 2.2 is the following comparison result.

Lemma 2.3. *Suppose that $u \in C^2(\overline{D})$ is a super-solution (sub-solution) to (2.5) with $\lambda_1(-a) < 0$, $u > 0$ in D and $u|_{\partial D} = 0$. Then $u(x) \geq (\leq)\ \theta_a(x)$.*

Proof. This follows from some standard super- and sub-solution arguments and the fact that (2.5) has at most one positive solution. We omit the details. #

Next we set up the fixed point index machinery for later use. Set $E = C_0(\overline{D}) \oplus C_0(\overline{D})$ and let P be the natural positive cone of E. Following [9], for $y \in P$, set $P_y = \big\{x \in E :\ y + tx \in P \text{ for some } t > 0\big\}$ and $S_y = \big\{x \in \overline{P}_y :\ -x \in \overline{P}_y\big\}$. Let y_0 be a fixed point of some compact operator $T : P \to P$, and denote by L the Fréchet derivative of T at y_0. We say L has property α at y_0 if there exist $t \in (0,1)$ and $w \in \overline{P}_{y_0} \setminus S_{y_0}$ such that $w - tLw \in S_{y_0}$. Next we state a general result of Dancer [9] and [15] on the fixed point index with respect to the positive cone P (see also [22]).

Theorem 2.1. 1) *If $I - L$ is invertible on E, and L has property α on $\overline{P}_{y_0}$, then $ind_P(T, y_0) = 0$.*

2) *If $I - L$ is invertible on E, and L does not have property α on $\overline{P}_{y_0}$, then $ind_P\big(T, y_0\big) = (-1)^\sigma$, where σ is the sum of the algebraic multiplicities of the eigenvalues of L which are greater than* 1.

3) *If $I - L$ is not invertible on E but $Ker(I-L) \cap \bar{P}_{y_0} = \{0\}$, then $ind_P(T, y_0) = 0$.*

Set $\Omega = \big\{(u, v) \in P : u \leq a,\ v \leq d + ca\big\}$, and define $A_t : \Omega \to P$ by

$$A_t\begin{pmatrix} u \\ v \end{pmatrix} = \big(-\Delta + K\big)^{-1}\begin{pmatrix} u\big(a + K - u - bv/(1 + tmu)\big) \\ v\big(d + K - v + cu/(1 + tmu)\big) \end{pmatrix}, \tag{2.6}$$

where $t \in [0,1]$ and $K = \max\big\{ac, b(d + ac)\big\}$. It follows from standard elliptic regularity theory that A_t is a completely continuous operator. A simple application of Lemma 2.3 shows that if (u, v) is a nonnegative solution of (1.1) with m replaced by tm, then

$$0 \leq u \leq \theta_a < a, \quad 0 \leq v \leq \theta_{d+ac} < d + ac. \tag{2.7}$$

Hence (u, v) is a solution of (1.1) with m replaced by tm in P if and only if it is a fixed point of A_t in Ω.

If $a > \lambda_1$ and $d > \lambda_1$, then $(0,0)$, $(\theta_a, 0)$ and $(0, \theta_d)$ are the only nonnegative fixed points of A_t which are not positive. Their fixed point indices are calculated in the following lemma.

Lemma 2.4. *Let $a > \lambda_1$ and $d > \lambda_1$. Then for any $t \in [0, 1]$,*

1) $ind_P(A_t, \Omega) = 1$;
2) $ind_P(A_t, (0, 0)) = ind_P(A_t, (\theta_a, 0)) = 0$;
3) $ind_P(A_t, (0, \theta_d)) = 1$ *if* $a < \lambda_1(b\theta_d)$*;* $= 0$ *if* $a > \lambda_1(b\theta_d)$.

Proof. By (2.7), A_t has no fixed point on $\partial\Omega$. It then follows from the homotopy invariance of the fixed point index that $ind_P(A_t, \Omega) \equiv constant$. Note that when $t = 0$, then A_0 is the abstract operator for the classical predator-prey system. By [10], we have $ind_P(A_0, \Omega) = 1$. Hence $ind_P(A_t, \Omega) = 1$. As in [10], [11] or [22], by Theorem 2.1, we can obtain $ind_P\big(A_t, (0, 0)\big) = ind_P\big(A_t, (\theta_a, 0)\big) = 0$ and $ind_P\big(A_t, (0, \theta_d)\big) = 1$ if $a < \lambda_1(b\theta_d)$; =0 if $a > \lambda_1(b\theta_d)$. Note that in [10], [11] or [22], only the case $m = 0$ is considered, but their calculations of the local indices work for all $m \geq 0$. #

When $d \leq \lambda_1$ and $a > \lambda_1$, A_t has only two nonnegative fixed points which are not positive, namely, $(0, 0)$ and $(\theta_a, 0)$. There is a result corresponding to Lemma 2.4 for the case $d \leq \lambda_1$. Its proof is similar to Lemma 2.4 and hence is omitted.

Lemma 2.5. *Let $a > \lambda_1$ and $d \leq \lambda_1$. Then for any $t \in [0, 1]$,*

1) $ind_P(A_t, \Omega) = 1$;
2) $ind_P(A_t, (0, 0)) = 0$;
3) $ind_P(A_t(\theta_a, 0)) = 0$ *if* $d > \lambda_1(-\frac{c\theta_a}{1+tm\theta_a})$*; =1 if* $d < \lambda_1(-\frac{c\theta_a}{1+tm\theta_a})$.

The following result will be repeatedly used in the later sections. It reduces the proof of uniqueness to that of non-degeneracy and linearly stability of all possible positive solutions.

Lemma 2.6. *Suppose that $a > \lambda_1$, and that $d_0 < \lambda_1$, $d_1 > \lambda_1$ are uniquely determined by*

$$d_0 = \lambda_1(-\frac{c\theta_a}{1 + m\theta_a}), \quad a = \lambda_1(b\theta_{d_1}).$$

Suppose further that $d \in (d_0, d_1)$ and that any positive solution (u, v) of (1.1) is non-degenerate and linearly stable, that is, the linearized eigenvalue problem

$$\begin{cases} \Delta h + \Big(a - 2u - \dfrac{bv}{(1+mu)^2}\Big)h - \dfrac{buk}{1+mu} + \eta h = 0 \quad \textit{in } D, \quad h|_{\partial D} = 0, \\ \Delta k + \Big(d - 2v + \dfrac{cu}{1+mu}\Big)k + \dfrac{cvh}{(1+mu)^2} + \tau\eta k = 0 \quad \textit{in } D, \quad k|_{\partial D} = 0 \end{cases} \tag{2.9}$$

has no eigenvalue η with $Re\eta \leq 0$. Then (1.1) has a unique positive solution and it is asymptotically stable.

Proof. By [18], we need only to prove the uniqueness. By assumption all of the positive solutions of (1.1) are nondegenerate. Since $d_0 < d < d_1$, it is also easy to show that the trivial and semitrivial nonnegative solutions are bounded away from the positive solutions. Hence it follows from a simple compactness argument that there are at most finitely many positive solutions. Let them be $\{(u_i, v_i) : 0 \leq i \leq l\}$ where $l \geq 0$. Using Theorem 2.1 and the nondegeneracy and stability of (u_i, v_i) one can easily show that $ind_P(A_1, (u_i, v_i)) = 1$. We leave the details of the proof to this statement to the interested reader, as they are very similar to the calculations in [10], [11].

Since $d_0 < d < d_1$, it follows from Lemmas 2.4 and 2.5 that $ind_P(A_1,(0,0)) = ind_P(A_1,(\theta_a,0)) = 0$, and that if $d > \lambda_1$, then $ind_P(A_1,(0,\theta_d)) = 0$. In other words, the sum of the fixed point indices of all the trivial and semitrivial fixed points of A_1 is zero. Hence by the additivity of the fixed point index, we have

$$1 = ind_P(A_1,\Omega) = \sum_{i=1}^{l} ind_P(A_1,(u_i,v_i)) + 0 = l.$$

This proves the uniqueness.#

Finally we consider the elliptic equation (1.4), which acts as a limiting problem of (1.1) when $m \to \infty$.

Lemma 2.7. *The problem (1.4) has a positive solution if and only if $\lambda_1 < a < \lambda_1(b\theta_d)$. Moreover, all positive solutions of (1.4) are unstable. Furthermore, there is some ϵ_1 small such that if $a \in (\lambda_1,\lambda_2] \cup [\lambda_1(b\theta_d)-\epsilon_1,\lambda_1(b\theta_d))$, then (1.4) has at most one positive solution and it is non-degenerate (if it exists).*

Proof. Suppose u is a positive solution of (1.4). Then

$$\lambda_1(-a + b\theta_d/(1+u)) = 0.$$

Since $0 < 1/(1+u) < 1$, we obtain

$$\lambda_1(-a) < \lambda_1(-a + b\theta_d/(1+u)) = 0 < \lambda_1(-a + b\theta_d).$$

Therefore $a \in \big(\lambda_1,\lambda_1(b\theta_d)\big)$. On the other hand, if $a \in \big(\lambda_1,\lambda_1(b\theta_d)\big)$, we show that (1.4) has at least a positive solution. To this end, we first prove that for any $\epsilon > 0$ small, there exists $C = C(\epsilon) > 0$ such that $\|u\|_{C^1} \le C$ for any positive solution of (1.4) with $a \ge \lambda_1 + \epsilon$. Suppose this is not true. Then we may assume that there exists some $\epsilon_0 > 0$, $a_i \to a \ge \lambda_1 + \epsilon_0$, u_i solutions of (1.4) with $a = a_i$ and $\|u_i\|_\infty \to \infty$, $1/(1+u_i) \to h$ weakly in L^2. Set $\hat u_i = u_i/\|u_i\|_\infty$. Then

$$-\Delta \hat u_i = \hat u_i\big(a_i - b\theta_d/(1+u_i)\big) \ \text{ in } \ D, \ \ \|\hat u_i\|_\infty = 1 \ \text{ in } \ D, \ \ \hat u_i|_{\partial D} = 0.$$

By standard elliptic regularity theory, we may assume $\hat u_i \to \hat u \ge 0$ in C^1, and $\hat u$ satisfies

$$-\Delta \hat u = \hat u(a - b\theta_d h) \ \text{ in } \ D, \ \ \|\hat u\|_\infty = 1, \ \ \hat u|_{\partial D} = 0.$$

Since $0 \le h \le 1$, the Harnack inequality is applicable, and we obtain $\hat u > 0$ in D. Then $1/(1+u_i) = 1/\big(1 + \|u_i\|_\infty \hat u_i\big) \to 0$ in L^2. Hence $h = 0$ and $-\Delta \hat u = a\hat u$. Since $a \ge \lambda_1 + \epsilon_0$, we have a contradiction. This establishes the desired a priori estimate.

By the global bifurcation theorem of Rabinowitz [27], we can easily show the existence of at least a positive solution to (1.4). However, for later purposes, we use a degree approach.

Set $\tilde P = \big\{u \in C_0^1(\overline D),\ u \ge 0\big\}$ and define $C_t : \tilde P \to \tilde P$ as

$$C_t(u) = \big(-\Delta + bd\big)^{-1}\big(u(t + bd - b\theta_d/(1+u)\big).$$

By virtue of our a priori estimate and the homotopy invariance property of the fixed point index, we obtain $ind_{\tilde P}(C_t,\tilde\Omega) \equiv constant$ for all $t \ge \lambda_1 + \epsilon$, where $\tilde\Omega$ is given by $\tilde\Omega = \big\{u \in \tilde P : \|u\|_\infty \le 2C(\epsilon)\big\}$. Since $u = 0$ is stable when $a < \lambda_1(b\theta_d)$ and unstable when $a > \lambda_1(b\theta_d)$, hence

$$ind_{\tilde P}(C_t,0) = \begin{cases} 1 & \text{if } t < \lambda_1(b\theta_d), \\ 0 & \text{if } t > \lambda_1(b\theta_d). \end{cases}$$

Since (1.4) has a unique non-negative solution $u \equiv 0$ if $a > \lambda_1(b\theta_d)$, then for any $t \geq \lambda_1 + \epsilon$, $ind_{\tilde{P}}(C_t, \tilde{\Omega}) \equiv 0$. Hence for $\lambda_1 + \epsilon \leq a \leq \lambda_1(b\theta_d)$, there exists $r > 0$ small such that

$$ind_{\tilde{P}}(C_a, \tilde{\Omega} \setminus B_r) = ind_{\tilde{P}}(C_a, \tilde{\Omega}) - ind_{\tilde{P}}(C_a, 0) = -1.$$

This shows that (1.4) has at least a positive solution.

Next we establish the uniqueness result. Consider the eigenvalue problem

$$-\Delta h + \big(-a + b\theta_d/(1+u)^2\big)h = \eta h \ \text{ in } \ D, \quad h|_{\partial D} = 0, \tag{2.10}$$

where u is a solution of (1.4). By (1.4) and the comparison principle of eigenvalues, we have

$$\lambda_1\big(-a + b\theta_d/(1+u)^2\big) < \lambda_1\big(-a + b\theta_d/(1+u)\big) = 0,$$

which also implies that any positive solution of (1.4) is unstable. On the other hand,

$$\lambda_2\big(-a + b\theta_d/(1+u)^2\big) > \lambda_2(-a) = \lambda_2 - a \geq 0$$

provided $a \leq \lambda_2$. Hence if $a \in (\lambda_1, \lambda_2]$, (2.10) has exactly one eigenvalue less than 0. Therefore for any positive solution u of (1.4), u is non-degenerate, unstable, and $ind_{\tilde{P}}(C_a, u) = -1$ provided $a \in (\lambda_1, \lambda_2]$. It is easy to show that (1.4) has at most finitely many positive solutions by the non-degeneracy of all non-negative solutions. If we denote all the positive solutions of (1.4) by $\{u_i, \ 1 \leq i \leq l\}$, then

$$0 = ind_{\tilde{P}}(C_a, \tilde{\Omega}) = ind_{\tilde{P}}(C_a, 0) + \sum_{i=1}^{l} ind_{\tilde{P}}(C_a, u_i) = l - 1,$$

which implies that (1.4) has a unique positive solution. The case that a is close to $\lambda_1(b\theta_d)$ follows from a similar argument, since all positive solutions of (1.4) approach zero as $a \to \lambda_1(b\theta_d)$. #

Remark 2.1. Problem (1.4) has properties very similar to those of problem (1.6) in [17]. In particular, as in [17], for any $a \in (\lambda_1, \lambda_1(b\theta_d))$, if the space dimension $n \geq 2$, then one can use domain variation arguments to show that, for a certain domain D, there are a large number of positive solutions of (1.4). This can be used as in [17] to show the last statement in Remark 1.3.

3. The case $d > \lambda_1$

In this section, we are mainly concerned with the case $d > \lambda_1$. Theorem A in the Introduction will follow as a consequence of the results in this section.

First let us recall that for $d > \lambda_1$, (1.1) has a positive solution if $a > \lambda_1(b\theta_d)$ (see [2], Theorem 5.1, or [3], Theorem 3.1). Moreover, it was observed in [3], page 426, that $a > \lambda_1(b\theta_d)$ is also necessary for (1.1) to have a positive solution if $m < (bd)^{-1}$. For general $m > 0$, it is easy to show that $a > \lambda_1(b\theta_d/(1 + m\theta_a))$ is a necessary condition (see [3], Theorem 3.1).

Throughout this section, unless otherwise specified, we shall always assume that $d > \lambda_1$ and let b, c and d be fixed. We shall use M, M_1 to denote generic positive constants depending on b, c and d, but *independent* of a; $M(\epsilon), M(\epsilon, \delta), M(\epsilon, A)$ will denote positive constants which may also depend on ϵ, δ and A in addition to b, c and d. Theorem A will follow from three more general theorems, the first of which concerns the case that m is large and a is bounded away from λ_1.

Theorem 3.1. *For any ϵ small, there exists $M = M(\epsilon)$ large such that for $m \ge M$,*

1) *if $a \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big)$, (1.1) has at least two positive solutions;*
2) *if $a \ge \lambda_1(b\theta_d)$, (1.1) has a unique positive solution and it is asymptotically stable.*

First of all, if $a \in [\lambda_1 + \epsilon, \infty)$, it is easy to find a positive solution for large m by the standard super-sub solution method. More precisely, we have

Lemma 3.1. *Given any $\epsilon > 0$ small, there exists $M = M(\epsilon)$ such that if $a \ge \lambda_1 + \epsilon$ and $m \ge M$, (1.1) has a positive solution $(\tilde{u}, \tilde{v})$ which satisfies*

$$\theta_{a-\epsilon/2} \le u \le \theta_a \quad \textit{and} \quad \theta_d \le v \le \theta_{d+\epsilon/2}. \tag{3.1}$$

Proof. Set $(\overline{u}, \overline{v}) = (\theta_a, \theta_{d+\epsilon/2})$ and $(\underline{u}, \underline{v}) = (\theta_{a-\epsilon/2}, \theta_d)$. By the super-sub solution method for predator-prey systems (see, e.g., [26] or [29]), it suffices to show that $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ are super-sub solutions to (1.1) for $m \ge M$, i.e, they satisfy

$$\begin{aligned} &\Delta\overline{u} + \overline{u}\big(a - \overline{u} - b\underline{v}/(1 + m\overline{u})\big) \le 0, \qquad \Delta\underline{u} + \underline{u}\big(a - \underline{u} - b\overline{v}/(1 + m\underline{u})\big) \ge 0, \\ &\Delta\overline{v} + \overline{v}\big(d - \overline{v} + c\overline{u}/(1 + m\overline{u})\big) \le 0, \qquad \Delta\underline{v} + \underline{v}\big(d - \underline{v} + c\underline{u}/(1 + m\underline{u})\big) \ge 0. \end{aligned} \tag{3.2}$$

It is trivial to see that $\overline{u}$ and $\underline{v}$ satisfy the corresponding inequalities in (3.2). By the definition of $\theta_{d+\epsilon/2}$, the inequality for $\overline{v}$ holds provided that $m \ge 2c/\epsilon$. For $\underline{u}$ to satisfy the inequality in (3.2), we only need to have

$$2b\theta_{d+\epsilon/2} \le \epsilon(1 + m\theta_{a-\epsilon/2}). \tag{3.3}$$

Since $a \ge \lambda_1 + \epsilon$, it suffices to have

$$2b\theta_{d+\epsilon/2} \le m\epsilon\theta_{\lambda_1+\epsilon/2}. \tag{3.4}$$

The inequality (3.4) holds as long as $m \ge M(\epsilon)$, where $M(\epsilon)$ is given by

$$M(\epsilon) = \max\Big\{\frac{2c}{\epsilon},\ \frac{2b}{\epsilon}\sup_D \frac{\theta_{d+\epsilon/2}}{\theta_{\lambda_1+\epsilon/2}}\Big\}. \tag{3.5}$$

Hence if $m \ge M(\epsilon)$ and $a \ge \lambda_1 + \epsilon$, then $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ are super-sub solutions to (1.1). This completes the proof of Lemma 3.1. #

In the following, the cases $a \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big)$ and $a \ge \lambda_1(b\theta_d)$ will be treated separately. We shall first establish the multiplicity result in Theorem 3.1. For this purpose, a property of the solution $(\tilde{u}, \tilde{v})$ in Lemma 3.1 is given in the following result.

Lemma 3.2. *For any $\epsilon > 0$ small and any $A > \lambda_1$, there exists $M = M(\epsilon, A) > 0$ large such that if $a \in (\lambda_1 + \epsilon, A\big]$ and $m \ge M$, then any positive solution which satisfies (3.1) is non-degenerate and linearly stable.*

Proof. If $a \in (\lambda_1 + \epsilon, A\big]$ and (u, v) satisfies (3.1), it is easy to see that (1.1) is a regular perturbation of (1.3) when m is large. Since (1.3) has a unique positive solution (θ_a, θ_d) which is also stable, Lemma 3.2 follows from a standard regular perturbation argument. We omit the details.#

Proof of 1), *Theorem* 3.1. For any $\epsilon > 0$ small, choose $M(\epsilon)$ and $M(\epsilon, \lambda_1(b\theta_d))$ as in Lemmas 3.1 and 3.2, and let $m \ge M = \max\big\{M(\epsilon),\ M(\epsilon, \lambda_1(b\theta_d))\big\}$. Suppose that for $m \ge M$ and $a \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big)$, (1.1) has a unique positive solution $(\tilde{u}, \tilde{v})$ as shown in Lemma 3.1. By virtue of Lemma 3.2, $I - L$ is invertible in E and L has no real eigenvalue greater than one, where L is the Fréchet derivative of A_1 at

$(\tilde{u}, \tilde{v})$. One can easily show that $S_{(\tilde{u},\tilde{v})} = \{0\}$. Hence L does not have property α. By Theorem 2.1 we obtain $ind_P\big(A_1, (\tilde{u}, \tilde{v})\big) = 1$. Then from Lemma 2.4 it follows that

$$\begin{aligned} 1 &= ind_P(A_1, \Omega) \\ &= ind_P\big(A_1, (\tilde{u}, \tilde{v})\big) + ind_P\big(A_1, (\theta_a, 0)\big) + ind_P\big(A_1, (0, \theta_d)\big) + ind_P\big(A_1, (0, 0)\big) \\ &= 2. \end{aligned}$$

This contradiction completes the proof. #

Remark 3.1. In fact, we can prove a little more than stated in part 1) of Theorem 1. That is, for $a \in (\lambda_1 + \epsilon, \lambda_1(b\theta_d))$, (1.1) has at least two positive solutions, with exactly one close to (θ_a, θ_d) and asymptotically stable, while all the others are close to $(0, \theta_d)$ and unstable. (Part of these will be proved later in Lemma 3.6.) Moreover, the number of positive solutions of the latter type is essentially determined by that of equation (1.4). These follow from arguments similar to those in [17], section 4.

The proof of part 2) of Theorem 3.1 is more difficult. First we show that there is no positive solution of (1.1) with small u component if $d \geq \lambda_1(b\theta_d)$ and m is large. For later purposes, we prove a little more than needed. We consider the following system with $t \in [0, 1]$:

$$(3.6t) \qquad \begin{cases} \Delta u + u\big(a - u - tbv/(1 + mu)\big) = 0 & \text{in } D, \ u|_{\partial D} = 0, \\ \Delta v + v\big(d - v + tcu/(1 + mu)\big) = 0 & \text{in } D, \ v|_{\partial D} = 0. \end{cases}$$

Lemma 3.3. *There exists M large such that, if $m \geq M$, then for any $a \geq \lambda_1(b\theta_d)$ and $t \in [0, 1]$ we have $u \geq \theta_{\tilde{\lambda}}$, where (u, v) is a positive solution of* (3.6t) *and* $\tilde{\lambda} = \big(\lambda_1 + \lambda_1(b\theta_d)\big)/2$.

Proof. Suppose that Lemma 3.3 fails. Then there exist $m_i \to \infty, a_i \geq \lambda_1(b\theta_d)$, $t_i \in [0, 1]$, and a positive solution to (3.6t_i) with $(a, m) = (a_i, m_i)$ such that $u_i \geq \theta_{\tilde{\lambda}}$ does not hold. We consider two possibilities here.

(i) $t_i \to t_0 \in [0, 1)$. Since $v_i \leq \theta_{d+c/m_i}$, then for i large we have

$$(3.7) \qquad -\Delta u_i \geq \big(\lambda_1(b\theta_d) - t_i b\theta_{d+c/m_i} - u_i\big)u_i \geq \big(\lambda_1(\bar{t}b\theta_d) - t_0 b\theta_d - u_i\big)u_i$$

for some $\bar{t} \in (t_0, 1)$. Therefore for large i, u_i is a super-solution to

$$(3.8) \qquad -\Delta w = \big(\lambda_1(\bar{t}b\theta_d) - t_0 b\theta_d - w\big)w \quad \text{in } D, \quad w|_{\partial D} = 0.$$

Due to the choice of $\bar{t}$, (3.8) has a unique positive solution w. Hence by Lemma 2.3, we have $u_i \geq w$ for all large i. Therefore

$$-\Delta u_i \geq \Big(\lambda_1(b\theta_d) - \frac{b}{m_i} \sup_D \frac{\theta_{d+c/m_i}}{w} - u_i\Big)u_i \geq (\tilde{\lambda} - u_i)u_i.$$

Then again by Lemma 2.3, we obtain $u_i \geq \theta_{\tilde{\lambda}}$ for large i, and this contradicts our assumption at the beginning. Hence the possibility (i) is ruled out.

(ii) $t_i \to 1$. Assume that $a_i \to a \in \big(\lambda_1(b\theta_d), \infty\big]$. We first tackle the case $a < \infty$. Let $\delta > 0$ be so small that $\lambda_1(b\theta_{d+\delta}) < \big(a + \lambda_1(b\theta_d)\big)/2$, which is possible as $a > \lambda_1(b\theta_d)$. For large i,

$$-\Delta u_i \geq u_i\big[(a + \lambda_1(b\theta_d))/2 - b\theta_{d+\delta} - u_i\big].$$

By Lemma 2.1 we obtain $u_i \geq \tilde{w}$, where $\tilde{w}$ is the unique positive solution to

$$-\Delta \tilde{w} = \tilde{w}\big((a + \lambda_1(b\theta_d))/2 - b\theta_{d+\delta} - \tilde{w}\big) \quad \text{in } D, \quad \tilde{w}|_{\partial D} = 0.$$

Then arguing the same as in the case (i), we reach the contradiction. The case $a = \infty$ can be treated similarly.

At last we consider the possibility $a_i \to \lambda_1(b\theta_d)$ and $t_i \to 1$. By standard elliptic regularity theory, we may also assume that $u_i \to u$, $v_i \to v$ in C^1 norm, and that $1/(1+m_i u_i) \to h$ weakly in L^2 with $0 \le h \le 1$. Therefore u satisfies the following equation weakly:

$$-\Delta u = u\big[\lambda_1(b\theta_d) - u - bvh\big],\ u|_{\partial D} = 0.$$

If $u \ge 0$ and $u \not\equiv 0$, by virtue of Harnack inequality, we have $u > 0$ in D. Thus $h \equiv 0$, $u = \theta_{\lambda_1(b\theta_d)}$, and then $u_i \to \theta_{\lambda_1(b\theta_d)}$ in C^1. Since $\theta_{\lambda_1(b\theta_d)} > \theta_{\tilde{\lambda}}$, this contradicts our assumption at the beginning of the proof. If $u \equiv 0$, we can also derive a contradiction. To this end, set $\hat{u}_i = u_i/\|u_i\|_\infty$. From $(3.6t_i)$ we see that $\hat{u}_i$ satisfies

$$\Delta \hat{u}_i + \hat{u}_i\big(a_i - u_i - bt_i v_i/(1+m_i u_i)\big) = 0,\quad \hat{u}_i|_{\partial D} = 0. \tag{3.9}$$

By virtue of standard regularity theory, we may assume $\hat{u}_i \to \hat{u}$ in C^1. From $u_i \to u \equiv 0$ it follows that $v_i \to \theta_d$ in C^1. Using this fact and $1/(1+m_i u_i) \to h$ weakly in L^2, by passing to the weak limit in (3.9) we obtain

$$\Delta \hat{u} + \hat{u}\big(\lambda_1(b\theta_d) - b\theta_d h\big) = 0,\quad \hat{u}|_{\partial D} = 0, \tag{3.10}$$

where $\|\hat{u}\|_\infty = 1$. By Harnack's inequality we have $\hat{u} > 0$. Let $\Psi > 0$ be the positive solution to

$$-\Delta\Psi + b\theta_d\Psi = \lambda_1(b\theta_d)\Psi,\ \Psi|_{\partial D} = 0,\ \max_{\overline{D}}\Psi = 1.$$

Multiplying (3.10) by Ψ and integrating, we obtain

$$\int_D \hat{u}\Psi\theta_d(h-1) = 0.$$

Since $0 \le h \le 1$, we must have $h = 1$ and $\hat{u} = \Psi$. In conclusion, $\hat{u}_i \to \Psi$ in C^1. Multiplying $(3.6t_i)$ with $(u,v,a,m) = (u_i, v_i, a_i, m_i)$ by Ψ and integrating by parts, after some rearrangement we obtain

$$\big[a_i - \lambda_1(b\theta_d)\big]\int_D u_i\Psi \le \int_D u_i^2\Psi + b\int_D \frac{u_i(v_i-\theta_d)\Psi}{1+m_i u_i} - bm_i\int_D \frac{u_i^2\Psi\theta_d}{1+m_i u_i}. \tag{3.11}$$

Here we need some estimate on $(v_i - \theta_d)/\|u_i\|_\infty$. Set $w_i = (v_i-\theta_d)/\|u_i\|_\infty$. Then using $(3.6t_i)$ we obtain

$$-\Delta w_i + (-d + 2\theta_d)w_i = w_i(\theta_d - v_i) + ct_i\hat{u}_i v_i/(1+m_i u_i)$$

Multiplying the above identity by w_i and integrating, we obtain

$$\begin{aligned}\lambda_1(-d+2\theta_d)\int_D w_i^2 &\le \int_D \big(|\nabla w_i|^2 + (-d+2\theta_d)w_i^2\big)\\ &\le c\|\hat{u}_i\|_\infty\|v_i\|_\infty\int_D w_i + \|v_i-\theta_d\|_\infty\int_D w_i^2\end{aligned} \tag{3.12}$$

Since $\|v_i - \theta_d\|_\infty \to 0$, and since $\|\hat{u}_i\|_\infty$ and $\|v_i\|_\infty$ are bounded, by the Hölder inequality we see that $\|w_i\|_2$ is bounded. Therefore by L^p estimates and Sobolev embedding theorems, shall we see that $\|w_i\|_\infty$ is bounded. In fact, by exploring the elliptic regularity further, using a compactness argument and the fact that $h = 1$, we can deduce that $w_i \to c(-\Delta - d + 2\theta_d)^{-1}(\Psi\theta_d)$ in the C^1 norm. Dividing (3.11) by $\|u_i\|_\infty^2$, since $\hat{u}_i \to \Psi$ in C^1, $1/(1+m_iu_i) \to 1$ weakly in L^2 and w_i are uniformly bounded, we find that

$$\frac{a_i - \lambda_1(b\theta_d)}{\|u_i\|_\infty}\int_D \hat{u}_i\Psi \le \int_D \hat{u}_i^2\Psi + b\int_D \frac{\hat{u}_i w_i\Psi}{1+m_iu_i} - bm_i\int_D \frac{\hat{u}_i^2\Psi\theta_d}{1+m_iu_i} \to -\infty$$

as $i \to \infty$. Hence $a_i - \lambda_1(b\theta_d)$ is negative if i is large, which contradicts our assumption that $a_i \ge \lambda_1(b\theta_d)$ for all i. #

By Lemma 3.3 and a simple variant of the proof of Lemma 3.2, we immediately obtain the following result.

Lemma 3.4. *Given any $A > \lambda_1(b\theta_d)$, there exists $M = M(A) > 0$ large such that if $a \in [\lambda_1(b\theta_d), A]$ and $m \ge M$, then any positive solution of (1.1) is non-degenerate and linearly stable.*

Next we consider the case that a is large. For later use in section 4, again we prove more than required now.

Lemma 3.5. *For any $\epsilon > 0$, there exists $A = A(\epsilon) > 0$ large (independent of d!) such that if $m \ge \epsilon, d \le \frac{1}{\epsilon}$ and $a \ge A$, then any positive solution of (1.1) is non-degenerate and linearly stable.*

Proof. Suppose the conclusion is false. Then we can find some $\epsilon_0 > 0, m_i \ge \epsilon_0, d_i \le \frac{1}{\epsilon_0}, a_i \to \infty, Re\eta_i \le 0$ and (h_i, k_i) smooth with $\|h_i\|_2^2 + \|k_i\|_2^2 = 1$ such that

$$\begin{cases} \Delta h_i + \big[a_i - 2u_i - \dfrac{bv_i}{(1+m_iu_i)^2}\big]h_i - \dfrac{bu_ik_i}{1+m_iu_i} + \eta_i h_i = 0, & h_i|_{\partial D} = 0, \\ \Delta k_i + \big[d_i - 2v_i - \dfrac{cu_i}{1+m_iu_i}\big]k_i + \dfrac{cv_ih_i}{(1+m_iu_i)^2} + \tau\eta_i k_i = 0, & k_i|_{\partial D} = 0, \end{cases} \tag{3.13}$$

where (u_i, v_i) is a positive solution to (1.1) with $(a, d, m) = (a_i, d_i, m_i)$. By [2] it is necessary that $d_i > \lambda_1 - c/m_i$ for (1.1) with $(a, d, m) = (a_i, d_i, m_i)$ to possess a positive solution. Hence we may assume $d_i \to d \in \big[\lambda_1 - c/\epsilon_0, 1/\epsilon_0\big]$ and $m_i \to m \in [\epsilon_0, \infty]$.

Next we show that $\|h_i\|_2 \to 0$. By Kato's inequality,

$$\begin{aligned} -\Delta|h_i| &\le -Re\big(\overline{h_i}/|h_i|\Delta h_i\big) \\ &\le \big[a_i - 2u_i - bv_i/(1+m_iu_i)^2\big]|h_i| + bu_i|k_i|/(1+m_iu_i) + Re\eta_i|h_i| \\ &\le (a_i - 2u_i)|h_i| + b|k_i|/m_i. \end{aligned} \tag{3.14}$$

Set $\delta = b(1+c)/\epsilon_0$. Since $v_i \le \theta_{d_i + c/m_i} \le (1+c)/\epsilon_0$, then

$$-\Delta u_i \ge u_i(a_i - bv_i - u_i) \ge u_i(a_i - \delta - u_i).$$

By Lemma 2.3, we find $u_i \ge \theta_{a_i - \delta}$. Multiplying (3.14) by $|h_i|$ and integrating by parts, we obtain

$$\begin{aligned} \lambda_1(-a_i + 2\theta_{a_i-\delta})\int_D |h_i|^2 &\le \lambda_1(-a_i + 2u_i)\int_D |h_i|^2 \\ &\le \int_D |\nabla|h_i||^2 + (-a_i + 2u_i)|h_i|^2 \\ &\le b/m_i \int_D |h_i||k_i| \le C \end{aligned} \tag{3.15}$$

for some positive constant C. By (2.4) in Lemma 2.2,

$$\lambda_1(-a_i + 2\theta_{a_i-\delta}) = -a_i + \lambda_1(2\theta_{a_i-\delta}) \geq -a_i + k_0(a_i - \delta) \to +\infty \tag{3.16}$$

as $i \to \infty$. From (3.15) and (3.16) we obtain $\|h_i\|_2 \to 0$. From (3.13) it follows that

$$\int_D |\nabla h_i|^2 = \int_D \big[a_i - 2u_i - \frac{bv_i}{(1+m_iu_i)^2}\big]|h_i|^2 - b\int_D \frac{u_i\overline{h_i}k_i}{1+m_iu_i} + \eta_i \int_D |h_i|^2,$$
$$\int_D |\nabla k_i|^2 = \int_D \big[d - 2v_i + \frac{cu_i}{1+m_iu_i}\big]|k_i|^2 + c\int_D \frac{v_i\overline{k_i}h_i}{(1+m_iu_i)^2} + \tau\eta_i \int_D |k_i|^2.$$

Adding the above two identities, we obtain

$$\eta_i = \int_D |\nabla h_i|^2 - \int_D \big[a_i - 2u_i - \frac{bv_i}{(1+m_iu_i)^2}\big]|h_i|^2 + b\int_D \frac{u_i\overline{h_i}k_i}{1+m_iu_i}$$
$$+ \tau^{-1}\Big\{\int_D |\nabla k_i|^2 - \int_D \big[d - 2v_i + \frac{cu_i}{1+m_iu_i}\big]|k_i|^2 - c\int_D \frac{v_i\overline{k_i}h_i}{(1+m_iu_i)^2}\Big\}.$$

It is easy to see that the imaginary part of the right hand side of the above identity is bounded, hence $Im\eta_i$ is bounded. On the other hand, since a_i is unbounded, we also need (3.15), (3.16) and the fact that $\int_D |\nabla|h_i||^2 \leq \int_D |\nabla h_i|^2$ to show that $Re\eta_i$ is bounded from below. Thus η_i is bounded as we assume $Re\eta_i \leq 0$. Since $\theta_{a_i-\delta} \leq u_i \leq \theta_{a_i}$ and $\theta_{a_i}/a_i \to 1$ uniformly in any compact subset of D as $i \to \infty$, then $v_i \to \theta_{d+c/m}$ in the C^1 norm. By an L^p estimate, $\|k_i\|_{W^{2,2}}$ is bounded. Hence we may assume $k_i \to k$ in H_0^1 strongly. By letting $i \to \infty$, we see that k satisfies the following equation weakly (then strongly):

$$\Delta k + \big(d + c/m - 2\theta_{d+c/m}\big)k + \tau\eta k = 0, \quad k|_{\partial D} = 0$$

with $Re\eta \leq 0$. The self-adjointness of the above problem implies that η is real. Furthermore, since $\|k\|_2 = 1$ and $d + \frac{c}{m} \geq \lambda_1$, we must have $d + \frac{c}{m} = \lambda_1, \eta = 0$ and $k = \alpha\Phi_1/\|\Phi_1\|_2$ with $|\alpha| = 1$. Again by virtue of Kato's inequality,

(3.17)
$$-\Delta|k_i| \leq -Re\big(\overline{k_i}/|k_i|\Delta k_i\big)$$
$$\leq \big[d_i + cu_i/(1+m_iu_i) - 2v_i\big]|k_i| + cv_i|h_i|/(1+m_iu_i)^2 + \tau Re\eta_i|k_i|.$$

Multiplying (3.17) by v_i and integrating, we obtain, after some rearrangement, that

$$\int_D v_i^2|k_i| \leq c\int_D v_i^2|h_i|/(1+m_iu_i)^2 + \tau Re\eta_i \int_D |k_i|v_i \leq c\int_D |h_i|v_i^2. \tag{3.18}$$

Since $d + c/m = \lambda_1$, using the equation for v_i divided by $\|v_i\|_\infty$, we can easily show that $v_i/\|v_i\|_\infty \to \Phi_1$ in C^1. Dividing both sides of (3.18) by $\|v_i\|_\infty^2$, we find that

$$\int_D \big(v_i/\|v_i\|_\infty\big)^2|k_i| \leq c\int_D \big(v_i/\|v_i\|_\infty\big)^2|h_i| \to 0 \tag{3.19}$$

since $\|h_i\|_2 \to 0$. However, the left hand side of (3.19) goes to $\int_D \Phi_1^3/\|\Phi_1\|_2$ as $i \to \infty$. This contradiction completes the proof. #

Proof of 2), *Theorem* 3.1. By Lemmas 3.4, 3.5 and [18], it suffices to show the uniqueness. This follows from Lemma 2.6 for $a > \lambda_1(b\theta_d)$. In order to include the case that $a = \lambda_1(b\theta_d)$, we use a slightly different approach.

Set $S = \big\{(u,v) \in E : \theta_{\tilde{\lambda}}/2 < u < a, \theta_d/2 < v < d + ca\big\}$ and define $B_t : \bar{S} \to P$ by

$$B_t \begin{pmatrix} u \\ v \end{pmatrix} = \big(-\Delta + K\big)^{-1} \begin{pmatrix} u\big(a + K - u - tbv/(1+mu)\big) \\ v\big(d + K - v + tcu/(1+mu)\big) \end{pmatrix}, \tag{3.20}$$

where $t \in [0,1]$ and $K = \max\big\{ac, b(d+ac)\big\}$. It follows from standard regularity results that B_t is a completely continuous operator. Clearly (u,v) is a solution of (3.7t) in S if and only if it is a fixed point of B_t. For $m \geq M$ and $a \geq \lambda_1(b\theta_d)$, B_t has no fixed point on ∂S as shown in Lemma 3.3. Therefore $ind_P(B_t, S) \equiv constant$. In particular, $ind_P(B_1, S) = ind_P(B_0, S)$. It is easy to show that B_0 has a unique fixed point (θ_a, θ_d) in S and $ind_P\big(B_0, (\theta_a, \theta_d)\big) = 1$. Hence $ind_P(B_1, S) = 1$.

From Lemmas 3.3–3.5, we know that for $m \geq M$, all fixed points of B_1 fall into S and they are non-degenerate and linearly stable. Then by a compactness argument it is easy to show that there are at most finitely many fixed points, which we denote by $\big\{(u_i, v_i)\big\}_{i=1}^k$. As showing $ind_P\big(A_1, (\tilde{u}, \tilde{v})\big) = 1$ in the proof of part 1) of Theorem 3.1, we can prove $ind_p(B_1, (u_i, v_i)) = 1$ for each i by Theorem 2.1 and Lemmas 3.4, 3.5. By virtue of the additivity property of the fixed point index, we have

$$k = \sum_{i=1}^k ind_P\big(B_1, (u_i, v_i)\big) = ind_P(B_1, S) = 1.$$

Hence for $m \geq M$ and $a \geq \lambda_1(b\theta_d)$, (1.1) has a unique positive solution and it is stable.#

Our next task is to establish the exact multiplicity and stability results for m large and a close to $\lambda_1 + \epsilon$ or $\lambda_1(b\theta_d)$. Let ϵ_1 be defined by Lemma 2.7 and define $\epsilon_0 = \min\{\lambda_2, \lambda_1(b\theta_d) - \epsilon_1/2\} - \lambda_1$. We have

Theorem 3.2. *For any $\epsilon \in (0, \epsilon_0)$, we can find $M = M(\epsilon)$ large such that if $a \in [\lambda_1 + \epsilon, \lambda_1 + \epsilon_0] \cup [\lambda_1(b\theta_d) - \epsilon_1, \lambda_1(b\theta_d))$ and $m \geq M$, (1.1) has exactly two positive solutions, one asymptotically stable and the other unstable.*

To prove Theorem 3.2, we need some intermediate results. In the following lemma, we shall show that (1.1) has only two types of positive solutions for m large and $a \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big)$. More precisely, we have

Lemma 3.6. *For any $\epsilon, \delta > 0$ small, there exists $M = M(\epsilon, \delta) > 0$ large such that if $m \geq M$ and $a \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big)$, we have either (a) $\|u - \theta_a\|_{C^1} + \|v - \theta_d\|_{C^1} \leq \delta$ or (b) $\|u\|_{C^1} + \|v - \theta_d\|_{C^1} \leq \delta$, where (u,v) is any positive solution of (1.1). Furthermore, if (b) occurs, by choosing $M(\epsilon, \delta)$ suitably larger, we have $\|mu - w\|_{C^1} \leq \delta$, where w is a positive solution of (1.4).*

Proof. Suppose that the conclusion is not true. Then there exist $m_i \to \infty, a_i \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big)$, and a solution (u_i, v_i) of (1.1) with $(a, m) = (a_i, m_i)$ such that (u_i, v_i) are bounded away from both (θ_a, θ_d) and $(0, \theta_d)$. We may assume $a_i \to a \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big]$ and $1/(1 + m_i u_i) \to h$ weakly in L^2 with $0 \leq h \leq 1$. It is easy to show that $v_i \to \theta_d$ in C^1. By virtue of an L^p estimate and the Sobolev embedding theorems, we may assume $u_i \to u$ in C^1 and u satisfies

$$\Delta u + (a - u - b\theta_d h)u = 0, \quad u \geq 0, \quad u|_{\partial D} = 0.$$

If $u \equiv 0$, then $(u_i, v_i) \to (u, \theta_d) = (0, \theta_d)$ in C^1, which contradicts our assumption that the (u_i, v_i) are bounded away from $(0, \theta_d)$; if $u \geq\not\equiv 0$, then by Harnack inequality we have $u > 0$ in D. Therefore $h = 0$ and $u = \theta_a$, which also contradicts the assumption. This completes the proof of the first part.

For the second part, it suffices to show that if $\|u_i\|_{C^1} + \|v_i - \theta_d\|_{C^1} \to 0, m_i \to \infty$ and $a_i \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big)$, then $m_i u_i$ is close to some positive solution of (1.4) with $a = a_i$ in the C^1 norm.

First we claim that $m_i\|u_i\|_\infty$ is uniformly bounded. If this is not true, we may assume $m_i\|u_i\|_\infty \to \infty$. Set $\tilde{u}_i = u_i/\|u_i\|_\infty$. Then

$$\Delta \tilde{u}_i + \tilde{u}_i\big(a_i - u_i - bv_i/(1 + m_i u_i)\big) = 0, \quad \|\tilde{u}_i\|_\infty = 1, \quad \tilde{u}_i|_{\partial D} = 0. \tag{3.21}$$

By standard elliptic regularity theory, we may assume $\tilde{u}_i \to \tilde{u}$ in the C^1 norm, $1/(1 + m_i u_i) \to h$ weakly in L^2 and $a_i \to a \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big]$. By passing to the limit in (3.21), we find that $\tilde{u}$ satisfies the following equation weakly:

$$\Delta \tilde{u} + (a - b\theta_d h)\tilde{u} = 0, \quad \|\tilde{u}\|_\infty = 1, \quad \tilde{u}|_{\partial D} = 0.$$

By Harnack's inequality, $\tilde{u} > 0$ in D. Since $m_i\|u_i\|_\infty \to \infty$ and $\tilde{u}_i \to \tilde{u}$, then $1/(1 + m_i u_i) = 1/(1 + m_i\|u_i\|_\infty \tilde{u}_i) \to 0$ in any compact subset of D. Therefore $h = 0$ and $a = \lambda_1$, which contradicts the assumption $a \geq \lambda_1 + \epsilon$. This proves our assertion.

Set $w_i = m_i u_i$. Then w_i satisfies

$$\Delta w_i + \big(a_i - u_i - bv_i/(1 + w_i)\big)w_i = 0, \ w_i|_{\partial D} = 0. \tag{3.22}$$

Since $\|w_i\|_\infty = m_i\|u_i\|_\infty$ is bounded, by standard elliptic regularity theory and Sobolev embedding theorems, we may assume $w_i \to w$ in C^1. Then by letting $i \to \infty$ in (3.22), we see that w is a non-negative solution of (1.4). There are two possibilities here:

(i) $a = \lambda_1(b\theta_d)$. In this case, $m_i u_i = w_i \to w \equiv 0$. Since any positive solution of (1.4) with $a = a_i$ is close to zero when $a_i \to \lambda_1(b\theta_d)$, $m_i u_i$ is certainly close to positive solutions of (1.4) with $a = a_i$.

(ii) $a < \lambda_1(b\theta_d)$. In this case, it suffices to show that w is a positive solution of (1.4). If not, by Harnack's inequality, we obtain $w \equiv 0$. Set $\tilde{w}_i = w_i/\|w_i\|_\infty$. Then

$$\Delta \tilde{w}_i + \big(a_i - u_i - bv_i/(1 + w_i)\big)\tilde{w}_i = 0, \quad \tilde{w}_i|_{\partial D} = 0. \tag{3.23}$$

Hence we may assume $\tilde{w}_i \to \tilde{w}$ in C^1. By passing to the limit in (3.23) we obtain

$$\Delta \tilde{w} + (a - b\theta_d)\tilde{w} = 0, \quad \tilde{w}|_{\partial D} = 0$$

as $w_i \to w \equiv 0$. Since $a < \lambda_1(b\theta_d)$, we must have $\tilde{w} \equiv 0$, which contradicts $\|\tilde{w}\|_\infty = \lim_{i\to\infty} \|\tilde{w}_i\|_\infty = 1$. This completes the proof.#

Lemma 3.7. *There exist $\epsilon_2 > 0$ small and $M_1 > 0$ large, both depending only on b, c and d, such that if $a \in \big[\lambda_1(b\theta_d) - \epsilon_2, \lambda_1(b\theta_d)\big)$ and $m \geq M_1$, then (1.1) has exactly two positive solutions, one asymptotically stable and the other unstable.*

Proof. First we show that for large m, (1.1) has a unique asymptotically stable positive solution of type (a) in Lemma 3.6. In fact, if we choose δ small enough in Lemma 3.6, then any positive solution of (1.1) of type (a) satisfies (3.1). Hence by Lemma 3.2, they are non-degenerate and linearly stable. Now by a simple variant of the proof of part 2) of Theorem 3.1, we find that there is only one positive solution of (1.1) satisfying (a) and it is asymptotically stable.

Next we show that (1.1) has a unique unstable positive solution of type (b). If we can prove this, then by Lemma 3.6, our proof of Lemma 3.7 is complete.

By Lemmas 2.7 and 3.6, if a solution (u, v) of (1.1) is close to $(0, \theta_d)$, then mu must be close to w, where w is the unique positive solution of (1.4). Hence to prove uniqueness, it suffices to show that for $a \in \big[\lambda_1(b\theta_d) - \epsilon_2, \lambda_1(b\theta_d)\big)$ and $m \geq M_1$, there is a unique pair (mu, v), (u, v) being a positive solution of (1.1), close to (w, θ_d) for certain ϵ_2 and M_1. Set $\hat{u} = mu$, $\mu = 1/m$, and consider

$$\text{(3.24)} \qquad \begin{cases} \Delta\hat{u} + \big(a - \mu\hat{u} - bv/(1+\hat{u})\big)\hat{u} = 0, & \hat{u}|_{\partial D} = 0, \\ \Delta v + (d - v + \mu c\hat{u}/(1+\hat{u}))v = 0, & \hat{v}|_{\partial D} = 0. \end{cases}$$

Clearly (u, v) solves (1.1) if and only if (mu, v) solves (3.24) with $\mu = 1/m$. Thus it suffices to prove uniqueness for (3.24). For fixed $\mu \geq 0$, regarding a as a parameter, we see that $\big(\lambda_1(b\theta_d), 0, \theta_d\big)$ is a simple bifurcation point of (3.24). By virtue of a variant of Theorem 1 of [7] (see, e.g., Theorem 5.3 in [4]), there exist $\delta_1 > 0$ and C^1 curves

$$\Gamma_\mu = \{\big(a(\mu, s), \hat{u}(\mu, s), v(\mu, s)\big) : 0 \leq s \leq \delta_1\},\ 0 \leq \mu \leq \delta_1,$$

such that, if $0 \leq \mu \leq \delta_1$, then all positive solutions of (3.24) close to $\big(\lambda_1(b\theta_d), 0, \theta_d\big)$ $= (a(0,0), \hat{u}(0,0), v(0,0))$ lie on the curve Γ_μ. Hence we need only show that these curves *uniformly* cover an a-range $\big[\lambda_1(b\theta_d) - \epsilon_2, \lambda_1(b\theta_d)\big)$ for suitably chosen ϵ_2, and for fixed μ, Γ_μ covers the range *only once*. It is easy to obtain

$$\frac{\partial a}{\partial s}(0,0) = -b\int_D \theta_d \Psi^2 \Big/ \int_D \Psi < 0.$$

By shrinking δ_1 we may assume that $\frac{\partial a}{\partial s}(\mu, s) < 0$ for $0 \leq \mu, s \leq \delta_1$. Hence

$$\lambda_1(b\theta_d) - a(0, \delta_1) = a(0,0) - a(0, \delta_1) > 0.$$

By the continuity of $a(\mu, s)$, there exists $\delta \in (0, \delta_1]$ such that

$$\epsilon_2 = \min_{0 \leq \mu \leq \delta}\big(\lambda_1(b\theta_d) - a(\mu, \delta_1)\big) > 0.$$

Therefore if $a \geq \lambda_1(b\theta_d) - \epsilon_2$, then $a(\mu, \delta_1) \leq a$ for any $\mu \in [0, \delta]$. This shows that for each $\mu \in [0, \delta]$, Γ_μ covers the a-range $\big[\lambda_1(b\theta_d) - \epsilon_2, \lambda_1(b\theta_d)\big)$. Moreover, since $(\partial a/\partial s)(\mu, s) \neq 0$ for $0 \leq \mu, s \leq \delta_1$, each curve covers the range only once. By choosing $M_1 = 1/\delta$, we see that for $m \geq M_1$ and $\lambda_1(b\theta_d) - \epsilon_2 \leq a < \lambda_1(b\theta_d)$, (1.1) has exactly one positive solution of type (b) in Lemma 3.6.

It remains to show that the positive solution of (1.1) close to $(0, \theta_d)$ is unstable. A simple calculation shows that η is an eigenvalue of the linearization of (1.1) at (u, v) with eigenfunction (h, k) if and only if it is an eigenvalue of the linearization of (3.24) with $\mu = 1/m$ at (mu, v) with eigenfunction (mh, k). Hence it suffices to show that the linearization of (3.24) has a negative eigenvalue at any point on the bifurcation curves Γ_μ obtained in the previous paragraph. This follows from a simple application of a variant of Theorem 1.16 of Crandall and Rabinowitz [8]. To be more precise, by Lemma 1.3 in [8], we can obtain a variant of Corollary 1.13 there:

There exist $\tau > 0$ and C^1 functions $\gamma : (\lambda_1(b\theta_d) - \tau, \lambda_1(b\theta_d) + \tau) \times (-\tau, \tau) \to R^1$ and $\beta : (-\tau, \tau) \times (-\tau, \tau) \to R^1$ such that $\gamma(a, \mu)$ is a simple eigenvalue of the linearization of (3.24) at $(a, 0, \theta_d)$ and $\beta(s, \mu)$ is a simple eigenvalue of the linearization of (3.24) at $(a, u, v) = (a(\mu, , s), \hat{u}(\mu, s), v(\mu, s))$ with $0 \leq \mu, s \leq \tau$.

Moreover, $\gamma(\lambda_1(b\theta_d), \mu) = \beta(0, \mu) = 0$. It is easy to check that, in fact, $\gamma(a, \mu) = \lambda_1(b\theta_d) - a$. Now by Theorem 1.16 in [8],

$$\lim_{s\to 0} \frac{-s\frac{\partial a}{\partial s}(0,s)\frac{\partial \gamma}{\partial a}(\lambda_1(b\theta_d),0)}{\beta(s,0)} = 1.$$

It follows that $\frac{\partial}{\partial s}\beta(0,0) = \frac{\partial}{\partial s}a(0,0) < 0$. Therefore, by shrinking δ_1 further, and using the continuity of the function β, we find that $\beta(s,\mu) < 0$ for all $0 \le s, \mu \le \delta_1$. This proves what we want.#

Proof of Theorem 3.2. By Lemma 3.7, it suffices to establish the exact multiplicity and stability when $a \in I \equiv [\lambda_1 + \epsilon, \lambda_1 + \epsilon_0] \cup [\lambda_1(b\theta_d) - \epsilon_1, \lambda_1(b\theta_d) - \epsilon_2]$ for any given $\epsilon \in (0, \epsilon_0)$, where ϵ_2 is as in Lemma 3.7.

From Lemma 3.6 we see that solutions of (1.1) for $a \in \big[\lambda_1 + \epsilon, \lambda_1(b\theta_d)\big)$ and m large are of two types, that is, types (a) or (b). As in the proof of Lemma 3.7, we can show that there is a unique asymptotically stable positive solution of type (a). Thus in order to complete the proof, we only need to show that there is a unique unstable positive solution of (1.1) close to $(0, \theta_d)$ if $a \in I$ and m is large. Again by Lemma 3.6, it suffices to prove that there is a unique unstable positive solution (u, v) of (1.1) such that (mu, v) is close to (w_a, θ_d), where w_a is the unique positive solution of (1.4) as shown in Lemma 2.7. In this connection, we consider (3.24) with $a \in I$ and μ small. Let $a^* \in I$. Since the unique solution w_{a^*} of (1.4) with $a = a^*$ is non-degenerate, then (w_{a^*}, θ_d) is a non-degenerate solution of (3.24) with $(a, \mu) = (a^*, 0)$. Clearly, (3.24) with $\mu > 0$ small is a regular perturbation of (3.24) with $\mu = 0$, and the perturbation is uniform for a in the compact set I. Thus it follows from the implicit function theorem that there exist δ, $\tilde{\epsilon} > 0$ small such that for any $a \in I$ and $0 \le \mu \le \tilde{\epsilon}$, (3.24) possesses a unique positive solution $(\hat{u}_a, v_a)$ which satisfies

$$\|\hat{u}_a - w_a\|_{C^1} + \|v_a - \theta_d\|_{C^1} \le \delta.$$

Set

$$M = \max\big\{1/\tilde{\epsilon},\ M(\epsilon, \delta)\big\},$$

where $M(\epsilon, \delta)$ is defined in Lemma 3.6. We see that for any $\epsilon \in (0, \epsilon_0)$, there exists $M = M(\epsilon)$ such that if $m \ge M$ and $a \in I$, then (1.1) has a unique positive solution of type (b).

It remains to establish the instability for the unique positive solution of (1.1) of type (b). Define T and $T_0 : C_0^{2,\alpha}(\bar{D}) \times C_0^{2,\alpha}(\bar{D}) \to C^\alpha(\bar{D}) \times C^\alpha(\bar{D})$ by

$$T\binom{h}{k} = \begin{pmatrix} \Delta h + [a - 2u - \frac{bv}{(1+mu)^2}]h - \frac{bu}{1+mu}k \\ -\Delta k + [d - 2v + \frac{cu}{1+mu}]k + \frac{cv}{(1+mu)^2}h \end{pmatrix}$$

and

$$T_0\binom{h}{k} = \begin{pmatrix} \Delta h + [a - \frac{b\theta_d}{(1+w_a)^2}]h \\ \Delta k + (d - 2\theta_d)k + \frac{c\theta_d}{(1+w_a)^2}h \end{pmatrix}.$$

It is easy to check that, as $m \to \infty$, $T \to T_0$ in the operator norm uniformly for (u, v) close to $(0, \theta_d)$ with mu close to w_a and $a \in [\lambda_1 + \epsilon, \lambda_1 + \epsilon_0]$. Since 0 belongs to the resolvent set of T_0 and

$$\eta_0 = \lambda_1(-a + b\theta_d/(1 + w_a)^2) < 0$$

is an eigenvalue of T_0, it follows from standard perturbation theory that 0 also belongs to the resolvent set of T and that T has an eigenvalue η close to η_0. In particular, $Re\eta < 0$. This shows that for all large m, the positive solution of (1.1) close to $(0, \theta_d)$ is non-degenerate and unstable. This completes the proof of Theorem 3.2. #

To conclude this section, we study in detail the case when $a \in (\lambda_1, \lambda_1 + \epsilon]$ and m is large. It is easy to see that Theorem A follows from Theorems 3.1, 3.2 and the following result.

Theorem 3.3. *There exist $\epsilon > 0$ small and $M > 0$ large, both depending only on b, c and d, such that for each $m \geq M$, we can find a unique $\tilde{a} \in (\lambda_1, \lambda_1 + \epsilon)$ so that if $a < \tilde{a}$, (1.1) has no positive solution; if $a = \tilde{a}$, (1.1) has a unique positive solution; if $a \in (\tilde{a}, \lambda_1 + \epsilon]$, (1.1) has exactly two positive solutions, one asymptotically stable and the other unstable. Furthermore, $\tilde{a} \to \lambda_1$ as $m \to \infty$.*

To establish Theorem 3.3, we need two technical lemmas.

Lemma 3.8. *If $a_i \to \lambda_1 +, m_i \to \infty$, and (u_i, v_i) is a degenerate positive solution of (1.1) with $(a, m) = (a_i, m_i)$, then $u_i \to 0, v_i \to \theta_d, u_i/\|u_i\|_\infty \to \Phi_1$ in C^1 and $m_i^q \|u_i\|_\infty \to \infty$ for any $q > 1/2$.*

Proof. It is trivial to show that $u_i \to 0$ and $v_i \to \theta_d$. Set $\hat{u}_i = u_i/\|u_i\|_\infty$. We have

$$\Delta \hat{u}_i + \big(a_i - u_i - bv_i/(1 + m_i u_i)\big)\hat{u}_i = 0, \quad \hat{u}_i|_{\partial D} = 0. \tag{3.25}$$

By elliptic regularity theory we may assume $\hat{u}_i \to \hat{u}$ in C^1 norm. Let $1/(1+m_i u_i) \to h$ weakly in L^2. Then by passing to the limit in (3.25) we obtain

$$\Delta \hat{u} + (\lambda_1 - b\theta_d h)\hat{u} = 0, \quad \hat{u} \geq\not\equiv 0, \quad \hat{u}|_{\partial D} = 0.$$

It follows that $h = 0$ and $\hat{u} = \Phi_1$. It remains to show that $m_i^q \|u_i\|_\infty \to \infty$ for any $q > 1/2$. Suppose that this is not true. Then instead of going to a subsequence we may assume that $m_i^{q_0} \|u_i\|_\infty \leq C$ for some $q_0 > 1/2$. First of all, we prove

$$\lim_{i \to \infty} m_i \|u_i\|_\infty (a_i - \lambda_1) = b \int_D \theta_d \Phi_1 \Big/ \int_D \Phi_1^2. \tag{3.26}$$

Multiplying the first equation of (1.1) with $(u, v, a, m) = (u_i, v_i, a_i, m_i)$ by $m_i \Phi_1$ and integrating, after some rearrangement we obtain

$$m_i \|u_i\|_\infty (a_i - \lambda_1) \int_D \hat{u}_i \Phi_1 = m_i \|u_i\|_\infty^2 \int_D \hat{u}_i^2 \Phi_1 + b \int_D v_i \Phi_1 m_i u_i/(1 + m_i u_i) \tag{3.27}$$

Since $m_i u_i/(1 + m_i u_i) \to 1$ in any compact subset of D and

$$m_i \|u_i\|_\infty^2 = \big(m_i^{q_0} \|u_i\|_\infty\big)^2 m_i^{1-2q_0} \leq C m_i^{1-2q_0} \to 0,$$

by passing to the limit in (3.27), we thus obtain (3.26).

Next we prove the claim that $\|u_i\|_\infty (a_i - \lambda_1)^{-2q_0} \to \infty$. If the claim fails, we may assume $\|u_i\|_\infty (a_i - \lambda_1)^{-2q_0} \leq C$. Since (u_i, v_i) is degenerate, there exists (h_i, k_i) satisfying (2.9) with $(u, v, a, m, \eta) = (u_i, v_i, a_i, m_i, 0)$ and $\|h_i\|_2 + \|k_i\|_2 = 1$. By virtue of elliptic regularity theory, we may assume $(h_i, k_i) \to (h, k)$ in the C^1 norm. Since $1/(1 + m_i u_i) \to 0$ on any compact subset of D, thus k satisfies

$$-\Delta k = (d - 2\theta_d)k, \quad k|_{\partial D} = 0.$$

Therefore we must have $k \equiv 0$, since $\lambda_1(2\theta_d) > d$, and then it is easy to see that $h_i \to \Phi_1/\|\Phi_1\|_2$ in C^1. Multiplying the equation for h_i by Φ_1 and integrating, after some rearrangement we find that

$$\begin{aligned}\int_D h_i\Phi_1 = 2\int_D h_i\hat{u}_i\Phi_1\frac{\|u_i\|_\infty}{a_i-\lambda_1} + b\int_D \frac{h_i\Phi_1\hat{u}_i}{1+m_iu_i}\frac{\|u_i\|_\infty}{a_i-\lambda_1} \\ + b\int_D \frac{h_iv_i\Phi_1}{(a_i-\lambda_1)(1+m_iu_i)^2}.\end{aligned} \tag{3.28}$$

The first and second term in the right hand side of (3.28) approach zero since

$$\|u_i\|_\infty/(a_i-\lambda_1) = \big[\|u_i\|_\infty(a_i-\lambda_1)^{-2q_0}\big](a_i-\lambda_1)^{2q_0-1} \le C(a_i-\lambda_1)^{2q_0-1} \to 0.$$

For the last term in (3.28), we have

$$\int_D \frac{h_iv_i\Phi_1}{(a_i-\lambda_1)(1+m_iu_i)^2} \le \int_D \frac{h_iv_i\Phi_1(a_i-\lambda_1)}{\hat{u}_i^2\big(m_i\|u_i\|_\infty(a_i-\lambda_1)\big)^2} \le C(a_i-\lambda_1) \to 0$$

by (3.26). Therefore by passing to the limit in (3.28), we obtain $\int_D h_i\Phi_1 \to 0$, which is impossible as $h_i \to \Phi_1/\|\Phi_1\|_2$. This proves our assertion.

Using Lemma 2.1, we have

$$\|u_i\|_\infty \le \|\theta_{a_i}\|_\infty \le C(a_i-\lambda_1).$$

Then from (3.26) we obtain $m_i \ge C(a_i-\lambda_1)^{-2}$. Hence by the above claim, we have

$$m_i^{q_0}\|u_i\|_\infty \ge C(a_i-\lambda_1)^{-2q_0}\|u_i\|_\infty \to \infty,$$

which contradicts the assumption that $m_i^{q_0}\|u_i\|_\infty$ is bounded. #

Let $(\hat{u},\hat{v})$ be a degenerate positive solution of (1.1) with $a=\hat{a}$, and let (h,k) be a solution of

$$\begin{cases} -\Delta h = \big(\hat{a}-2\hat{u}-b\hat{v}/(1+m\hat{u})^2\big)h - b\hat{u}k/(1+m\hat{u}), & h|_{\partial D}=0, \\ -\Delta k = \big(d+c\hat{u}/(1+m\hat{u})-2\hat{v}\big)k + c\hat{v}h/(1+m\hat{u})^2, & k|_{\partial D}=0 \end{cases} \tag{3.29}$$

with $\|h\|_2+\|k\|_2=1$. Let Z be a complement to the span of (h,k) in $(W^{2,p}\cap H_0^1) \times (W^{2,p}\cap H_0^1)$ with $p>1$.

Lemma 3.9. *There exist ϵ_1 small, depending only on b, c and d, and M_1 large such that if $\lambda_1 < \hat{a} \le \lambda_1+\epsilon_1, m \ge M_1$ and $(\hat{u},\hat{v})$ is a degenerate positive solution of (1.1) with $a=\hat{a}$, then (3.29) has a unique solution (h,k) up to a change of the sign, and positive solutions of (1.1) close to $(\hat{a},\hat{u},\hat{v})$ lie on a C^1curve given by*

$$\big(\hat{a}(s),\hat{u}(s),\hat{v}(s)\big) = \big(\hat{a}+s\tau(s), \hat{u}+sh+s\phi(s), \hat{v}+sk+s\psi(s)\big), \tag{3.30}$$

where $-\delta \le s \le \delta$ for some $\delta>0$, $\tau(0)=0$, $\tau'(0)>0$, $\phi,\psi : (-\delta,\delta) \to Z$ with $\big(\phi(0),\psi(0)\big)=(0,0)$. In particular, (1.1) has no positive solution near $(\hat{u},\hat{v})$ for a close to but less than $\hat{a}$.

Proof. Suppose that (u_i,v_i) are degenerate positive solutions of (1.1) with $(a,m) = (a_i,m_i)$, and $a_i \to \lambda_1, m_i \to \infty$. It suffices to show that, for all large i, (3.29) with $(\hat{a},m,u,v)=(a_i,m_i,u_i,v_i)$ has a unique solution (h_i,k_i) and that positive solutions (a,u,v) of (1.1) with $m=m_i$ close to (a_i,u_i,v_i) lie on a curve given by (3.30) with the subscript i added at the obvious places.

Let $p > 1$ and set $X_0 = W^{2,p} \cap H_0^1, X = X_0 \times X_0$, $Y = L^p \times L^p$. Hence $Y^* = L^q \times L^q$, where $1/p + 1/q = 1$. Define $T_i : X \to Y$ as

$$T_i \binom{h}{k} = \begin{pmatrix} \Delta h + \big(a_i - 2u_i - bv_i/(1+m_iu_i)^2\big)h - bu_ik/(1+m_iu_i) \\ \Delta k + \big(d + cu_i/(1+m_iu_i) - 2v_i\big)k + cv_ih/(1+m_iu_i)^2 \end{pmatrix}.$$

As shown in Lemma 3.8, we have $u_i \to 0, v_i \to \theta_d, u_i/\|u_i\|_\infty \to \Phi_1, m_i\|u_i\|_\infty \to \infty$. Therefore $T_i \to T_0$ in the operator norm, where T_0 is given by

$$T_0 \binom{h}{k} = \begin{pmatrix} \Delta h + \lambda_1 h \\ \Delta k + (d - 2\theta_d)k \end{pmatrix}.$$

It is easy to see that $N(T_0) =$ span of $(\Phi_1, 0)$, and zero is a K-simple eigenvalue of T_0, where K is the natural injection from X into Y. Hence for large i, there is a K-simple eigenvalue γ_i of T_i close to zero. Since(u_i, v_i) is degenerate, then zero is an eigenvalue of T_i. Therefore $\gamma_i = 0$ for large i as T_i has only one eigenvalue close to zero. Let (h_i, k_i) be the unique corresponding eigenvector of T_i with $\|h_i\|_2+\|k_i\|_2 = 1$. Clearly (h_i, k_i) is the unique solution of (3.29) with $(\hat{a}, m, u, v) = (a_i, m_i, u_i, v_i)$. As in the proof of Lemma 3.8, we may assume $(h_i, k_i) \to (\Phi_1/\|\Phi_1\|_2, 0)$. Since $(\Phi_1, 0) \notin R(T_0)$, it is easy to show that $(u_i, 0) \notin R(T_i)$ for all large i. Hence we can use Theorem 3.2 of [8] to obtain the required C^1 curve

$$\big(a_i(s), u_i(s), v_i(s)\big) = \big(a_i + s\tau_i(s), u_i + sh_i + s\phi_i(s), v_i + sk_i + s\psi_i(s)\big),$$

where $s \in (-\delta_i, \delta_i), \tau_i(0) = 0, (\phi_i(0), \psi_i(0)) = (0, 0)$ and $(\phi_i(s), \psi_i(s))$ is in the complement to the span of (h_i, k_i) in X.

It remains to show that $\tau_i^{'}(0) > 0$ for large i. By differentiating the equation of $u_i(s)$ with respect to s twice at $s = 0$, we obtain

$$\begin{aligned} -\Delta(2\phi_i^{'}(0)) &= a_i^{''}(0)u_i + 2a_i\phi_i^{'}(0) - 2h_i^2 - 4u_i\phi_i^{'}(0) - 2bv_i\phi_i^{'}(0)/(1+m_iu_i)^2 \\ &+ 2b\big[v_im_ih_i^2/(1+m_iu_i)^3 - h_ik_i/(1+m_ik_i)^2 - u_i\psi_i^{'}(0)/(1+m_iu_i)\big]. \end{aligned}$$

Similarly, by differentiating the equation of v_i we have

$$\begin{aligned} -\Delta(2\psi_i^{'}(0)) &= 2d\psi_i^{'}(0) - 2k_i^2 - 4v_i\psi_i^{'}(0) + 2cv_i\phi_i^{'}(0)/(1+m_iu_i)^2 \\ &- 2c\big[v_im_ih_i^2/(1+m_iu_i)^3 - h_ik_i/(1+m_ik_i)^2 - u_i\psi_i^{'}(0)/(1+m_iu_i)\big]. \end{aligned}$$

Hence we obtain

$$T_i \begin{pmatrix} 2\phi_i^{'}(0) \\ 2\psi_i^{'}(0) \end{pmatrix} = \begin{pmatrix} -a_i^{''}(0)u_i + 2h_i^2 + 2bh_ik_i/(1+m_iu_i)^2 - 2bm_ih_i^2v_i/(1+m_iu_i)^3 \\ 2k_i^2 + 2cm_iv_ih_i^2/(1+m_iu_i)^3 - 2ch_ik_i/(1+m_iu_i)^2 \end{pmatrix}.$$

Set $l_0(u, v) = \int_D u\Phi_1$. Then it is easy to check that $l_0 \in N(T_0^*)$ and $N(l_0) = R(T_0)$. Choose $l_i \in N(T_i^*)$ with $\|l_i\| = \|l_0\|$. Then we have $l_i \to l_0$ in Y^* if we chose l_i in the correct sign. By the characterization of Y^*, we can find $(f_i, g_i) \in Y^*$ such that

$$l_i(u, v) = \int_D f_iu + \int_D g_iv$$

for any $(u, v) \in Y$. Since $l_i \to l_0$, $f_i \to \Phi_1$ and $g_i \to 0$ in $L^q(D)$. Hence

$$\begin{aligned} 0 =& \langle l_i, T_i(2\phi_i^{'}(0), 2\psi_i^{'}(0))\rangle \\ =& \int_D f_i\big[-a_i^{''}(0)u_i + 2h_i^2 + 2bh_ik_i/(1+m_iu_i)^2 - 2bm_ih_i^2v_i/(1+m_iu_i)^3\big] \\ &+ \int_D g_i\big[2k_i^2 + 2cm_iv_ih_i^2/(1+m_iu_i)^3 - 2ch_ik_i/(1+m_iu_i)^2\big]. \end{aligned}$$

After some arrangement we obtain

$$
\begin{aligned}
a_i''(0)\int_D f_i u_i =& 2\int_D f_i h_i^2 + 2b\int_D \frac{f_i h_i k_i}{(1+m_i u_i)^2} - 2c\int_D \frac{h_i g_i k_i}{(1+m_i u_i)^2} \\
&+ 2\int_D g_i k_i^2 - 2b\int_D \frac{f_i m_i h_i^2 v_i}{(1+m_i u_i)^3} + 2c\int_D \frac{g_i m_i v_i h_i^2}{(1+m_i u_i)^3}.
\end{aligned} \tag{3.31}
$$

It is easy to show that the first term in the right hand side of (3.31) goes to $2\int_D \Phi_1^3/\int_D \Phi_1^2$, and the second, third and fourth term approach zero as $\|k_i\|_2 \to 0$. The fifth term also goes to zero, as shown by the following argument:

$$
\begin{aligned}
\int_D \frac{f_i m_i h_i^2 v_i}{(1+m_i u_i)^3} &= \frac{1}{m_i^2\|u_i\|^3}\int_D \frac{f_i h_i^2 v_i}{(u_i/\|u_i\|_\infty)^3}\frac{(m_i u_i)^3}{(1+m_i u_i)^3} \\
&\le \frac{1}{(m_i^{2/3}\|u_i\|_\infty)^3}\int_D \frac{f_i h_i^2 v_i}{(u_i/\|u_i\|_\infty)^3} \to 0
\end{aligned}
$$

since $m_i^{2/3}\|u_i\|_\infty \to \infty$ as established in Lemma 3.8. The last term can be treated similarly. By passing to the limit in (3.31), we obtain

$$
\lim_{i\to\infty} a_i''(0)\|u_i\|_\infty \int_D \Phi_1^2 = 2\int_D \Phi_1^3/\int_D \Phi_1^2.
$$

Hence $a_i''(0) \to \infty$ as $\|u_i\|_\infty \to 0$. In particular, we have $a_i''(0) > 0$ for all large i. Hence $\tau_i'(0) > 0$ for large i. This completes the proof of Lemma 3.9. #

Proof of Theorem 3.3. Fix $\epsilon > 0$ small such that $\epsilon < \min\{\epsilon_0, \epsilon_1\}$ and set $M = \max\{M_1, M(\epsilon)\}$, where ϵ_0 and $M(\epsilon)$ are given in Theorem 3.2, and ϵ_1, M_1 are as in Lemma 3.9. By Theorem 3.2, (1.1) has exactly two positive solutions for $m \ge M$ and $a \in [\lambda_1+\epsilon, \lambda_1+\epsilon_0]$. Set

$$
\tilde a = \inf\{a : a > \lambda_1,\ (1.1) \text{ has at least a positive solution}\}.
$$

By Theorem 3.1, clearly $\tilde a \to \lambda_1$ as $m \to \infty$. By the definition of $\tilde a$ and a simple compactness argument, there exist $\tilde u \ge 0$ and $\tilde v \ge 0$ such that $(\tilde u, \tilde v)$ is a solution of (1.1) with $a = \tilde a$. By virtue of the continuity and compactness argument, we have $\tilde v \ge \theta_d$ and

$$
\lambda_1\big(-\tilde a + \tilde u + b\tilde v/(1+m\tilde u)\big) = 0.
$$

If $\tilde u \equiv 0$, then $\tilde v \equiv \theta_d$ and $\tilde a \equiv \lambda_1(b\theta_d)$. This is impossible since $\tilde a \le \lambda_1+\epsilon < \lambda_1(b\theta_d)$. Hence $\tilde u \ge\not\equiv 0$, and by Harnack's inequality $\tilde u > 0$ in D. This implies that $\tilde a > \lambda_1$. Furthermore, $(\tilde u, \tilde v)$ must be a degenerate positive solution of (1.1). Otherwise we can apply the implicit function theorem to extend the solution of (1.1) to the left of $\tilde a$, which contradicts the definition of $\tilde a$. Hence we can apply Lemma 3.9 to conclude that positive solutions close to $(\tilde a, \tilde u, \tilde v)$ form a curve Γ which passes through $(\tilde a, \tilde u, \tilde v)$ and bends to the right of $\tilde a$. Again by Lemma 3.9, all positive solutions of (1.1) on Γ must be non-degenerate except for $(\tilde u, \tilde v)$ at $a = \tilde a$. Thus Γ can be extended till $a = \lambda_1 + \epsilon_1$ by the implicit function theorem.

We show next that for $a \in [\tilde a, \lambda_1+\epsilon]$, all positive solutions lie on Γ and its extension. Suppose not. Then

$$
\hat a = \inf\{a:\ a \le \lambda_1+\epsilon,\ (1.1) \text{ has a positive solution not on } \Gamma\}
$$

is well-defined. Repeating the same argument as for Γ, we see that there is another solution curve $\hat\Gamma$ which can also be extended up to $a = \lambda_1+\epsilon_1$. Hence $\hat\Gamma$ must join Γ at some $a \le \lambda_1+\epsilon$ by Theorem 3.2. But this is impossible since positive solutions on Γ and its extension are non-degenerate, except for $(\tilde{a}, \tilde{u}, \tilde{v})$, which is a simple turning point. This contradiction shows that there exists a unique $\tilde{a}$ as stated in Theorem 3.3.

It remains to establish the stability result. With m fixed, let T be defined as in the proof of Theorem 3.2. Then, as in the proof of Lemma 3.9, we can show that there exists a K-simple eigenvalue $r(T)$ for the operator T and that $r(T)$ depends continuously on (a, u, v) along Γ. It is well-known that the sign of $r(T)$ determines the stability of the positive solution (u, v). Since $(\tilde{a}, \tilde{u}, \tilde{v})$ is the only degenerate point on Γ and its extension, we find that, along the extended Γ, $r(T)$ can take the value zero only at $(\tilde{a}, \tilde{u}, \tilde{v})$. Indeed, $r(T)$ changes sign at this point, since at $a = \lambda_1 + \epsilon$ the two solutions on the extended Γ are known to be stable and unstable respectively. This proves our stability result. The proof of Theorem 3.3 is thus complete. #

4. The case $d \le \lambda_1$

In this section, we mainly consider the case $d \le \lambda_1$. The existence problem for this case is completely understood. In fact, the following result has been proved by Blat and Brown [2].

Proposition 4.1. 1) *If $d \le \lambda_1 - c/m$, then (1.1) has no positive solution;*

2) *if $\lambda_1 - c/m < d \le \lambda_1$, then there exists a unique constant a_0 defined as in Theorem* B *such that (1.1) has no positive solution for $a \le a_0$, and at least one positive solution for $a > a_0$.*

Remark 4.1. The result in section 5 of [2] does not include the case $d = \lambda_1$, but this case follows easily from Theorem 4.5 there.

Our purpose in this section is to better understand the number and stability of the positive solutions of (1.1) when $\lambda_1 - c/m < d \le \lambda_1$ and $a > a_0$. It turns out that both uniqueness and non-uniqueness can occur. In this case, quite interestingly, the size of bc plays an important role. We will prove Theorem B, which roughly says that uniqueness holds if bc is relatively small and m is large. Again our results are optimal when m is large. We will also construct examples to show that if bc becomes large and m is large, then non-uniqueness occurs.

Throughout this section, let b and c be fixed positive constants. We will first establish Theorem B by proving two general results.

Theorem 4.1. *Given any $\epsilon > 0$, there exists some constant M, depending only on ϵ, b and c, such that if $m \ge M$ and $a \ge \lambda_1 + \epsilon$, then for any $d \le \lambda_1$, (1.1) has at most one positive solution. Moreover, the positive solution (if it exists) is non-degenerate and asymptotically stable.*

Proof. By Proposition 4.1 and Lemma 2.6, it suffices to show the non-degeneracy and stability of the positive solutions. Note that here we use the fact that $a > a_0$ if and only if $d > d_0$. By Lemma 3.5, we only need to consider the case that a is not large. We assume that $a \in [\lambda_1 + \epsilon, A]$, where A is determined by Lemma 3.5.

We shall argue by contradiction. Suppose that for some $\epsilon > 0$, we can find $m_i \to \infty$, $a_i \in [\lambda_1 + \epsilon, A]$ and $d_i \in (\lambda_1 - c/m_i, \lambda_1]$ such that (1.1) with $(a, d, m) = (a_i, d_i, m_i)$ has a positive solution (u_i, v_i) which is either degenerate or linearly unstable. That is, there exist η_i with $Re\eta_i \le 0$, (h_i, k_i) smooth and $(h_i, k_i) \ne (0, 0)$

such that

$$
(4.1)\quad \begin{cases} \Delta h_i + \big[a_i - 2u_i - \dfrac{bv_i}{(1+m_iu_i)^2}\big]h_i - \dfrac{bu_ik_i}{1+m_iu_i} + \eta_i h_i = 0, \quad h_i|_{\partial D} = 0, \\ \Delta k_i + \big[d_i + \dfrac{cu_i}{1+m_iu_i} - 2v_i\big]k_i + \dfrac{cv_ih_i}{(1+m_iu_i)^2} + \tau\eta_i k_i = 0, \quad k_i|_{\partial D} = 0. \end{cases}
$$

It follows from Lemma 2.3 and the equations for u_i and v_i that

$$
0 \le v_i \le \theta_{d_i + c/m_i}, \quad \theta_{a_i - \delta_i} \le u_i \le \theta_{a_i},
$$

where $\delta_i = b\|\theta_{d_i+c/m_i}\|_\infty \to 0$ as $i \to \infty$. It is easy to see that $v_i \to 0$ in L^∞. If we assume that $a_i \to a$, then $u_i \to \theta_a$ in L^∞. We may let $\|h_i\|_2^2 + \|k_i\|_2^2 = 1$. Then it follows from (4.1) that

$$
\begin{aligned}
\eta_i = & \int_D |\nabla h_i|^2 - \int_D \big[a_i - 2u_i - \frac{bv_i}{(1+m_iu_i)^2}\big]|h_i|^2 + \int_D \frac{bu_i}{1+m_iu_i} k_i \bar{h}_i \\
& + \tau^{-1}\Big\{ \int_D |\nabla k_i|^2 - \int_D \big[d_i - 2v_i + \frac{cu_i}{1+m_iu_i}\big]|k_i|^2 - \int_D \frac{cv_i}{(1+m_iu_i)^2} h_i \bar{k}_i \Big\}.
\end{aligned}
$$

From the above identity, one easily sees that $\{Im\eta_i\}$ is bounded, and that $\{Re\eta_i\}$ is bounded from below. By the assumption $Re\eta_i \le 0$ we see that $\{\eta_i\}$ must be bounded. Without loss of generality, we assume that $\eta_i \to \eta$. Then $Re\eta \le 0$. By (4.1) and standard elliptic regularity theory, $\{h_i\}$ and $\{k_i\}$ are bounded in $W^{2,2}$, and thus we may assume that $h_i \to h$ and $k_i \to k$ in H_0^1. Passing to the weak limit in (4.1), we obtain

$$
\begin{cases} \Delta h + ah - 2\theta_a h + \eta h = 0, \quad h|_{\partial D} = 0, \\ \qquad \Delta k + \lambda_1 k + \tau\eta k = 0, \quad k|_{\partial D} = 0. \end{cases}
$$

Hence η is real. Since $\eta \le 0$ and $\lambda_1(2\theta_a) > a$, it follows from the equation for h that $h = 0$. Thus $k \ne 0$, and by the equation for k we obtain $\eta = 0$ and $k = \alpha\Phi_1/\|\Phi_1\|_2$ with $|\alpha| = 1$. Now by Kato's inequality,

$$
\begin{aligned}
-\Delta|k_i| \le & - Re(\frac{\bar{k}_i}{|k_i|}\Delta k_i) \\
\le & cv_i|h_i|/(1+m_iu_i)^2 + \big[d_i + cu_i/(1+m_iu_i) - 2v_i\big]|k_i| + \tau Re\eta_i |k_i|.
\end{aligned}
$$

Multiplying the above inequality by v_i and integrating over D, after a simple rearrangement we obtain

$$
\int_D v_i^2|k_i| \le \tau Re\eta_i \int_D |k_i|v_i + \int_D cv_i^2|h_i|/(1+m_iu_i)^2.
$$

Hence

$$
(4.2)\qquad \int_D \big(v_i/\|v_i\|_\infty\big)^2|k_i| \le c\int_D \big(v_i/\|v_i\|_\infty\big)^2|h_i|.
$$

On the other hand, if we let $\hat{v}_i = v_i/\|v_i\|_\infty$, then

$$
-\Delta\hat{v}_i = \hat{v}_i\big[d_i - v_i + cu_i/(1+m_iu_i)\big], \ \hat{v}_i|_{\partial D} = 0.
$$

Since the right hand side of the above equation is bounded in L^∞, by standard elliptic regularity theory and Sobolev embedding theorems, $\{\hat{v}_i\}$ is compact in $C^{1,\alpha}$ for some $\alpha \in (0,1)$. We may assume, choosing a subsequence if needed, that $\hat{v}_i \to \hat{v}$. Then passing to the limit in the above equation, we obtain

$$
-\Delta\hat{v} = \lambda_1\hat{v}, \quad \|\hat{v}\|_\infty = 1, \quad \hat{v} \ge 0, \quad \hat{v}|_{\partial D} = 0.
$$

Thus we must have $\hat{v} = \Phi_1$. Hence

$$\int_D \big(v_i/\|v_i\|_\infty\big)^2 |k_i| \to \int_D \Phi_1^3/\|\Phi\|_2 > 0.$$

On the other hand, by (4.2) and $\|h_i\|_2 \to \|h\|_2 \equiv 0$,

$$\int_D \big(v_i/\|v_i\|_\infty\big)^2 |k_i| \le c \int_D \big(v_i/\|v_i\|_\infty\big)^2 |h_i| \to 0.$$

This contradiction completes the proof.#

Theorem 4.1 says that if $d \le \lambda_1$ and a is bounded away from λ_1, then uniqueness follows if m is large. In the remaining part of this section, we will concentrate on the case that a is near λ_1. This turns out to be a rather delicate case. In particular, large m does not necessarily imply uniqueness, and the size of bc will play an important role.

Theorem 4.2. *Suppose that $bc \le 4$. Then there exists ϵ small, depending only on b and c, such that for any $a \in (\lambda_1, \lambda_1 + \epsilon]$, $d \le \lambda_1$ and $m \ge 0$, (1.1) has at most one positive solution. Moreover, the positive solution (if it exists) is asymptotically stable.*

Proof. By Proposition 4.1 and Lemma 2.6, it suffices to show that any positive solution of (1.1) is non-degenerate and linearly stable.

Again we argue indirectly. Suppose that we can find $a_i \to \lambda_1+$, $m_i \ge 0$ and $d_i \le \lambda_1$ such that (1.1) with $(a, d, m) = (a_i, d_i, m_i)$ has a positive solution (u_i, v_i) which is either degenerate or unstable. That is, there exist η_i with $Re\eta_i \le 0$, (h_i, k_i) smooth and $(h_i, k_i) \ne (0, 0)$ such that (4.1) holds.

Since $0 \le u_i \le \theta_{a_i}$, we have $u_i \to 0$ in L^∞. From the equations for v_i, we can easily deduce that

$$\theta_{d_i + c\|\theta_{a_i}\|_\infty} \ge v_i \ge \theta_{d_i}. \tag{4.3}$$

Hence from $v_i \not\equiv 0$ it follows that $d_i + c\|\theta_{a_i}\|_\infty > \lambda_1$. Thus

$$\lambda_1 - c\|\theta_{a_i}\|_\infty < d_i \le \lambda_1.$$

This implies $d_i \to \lambda_1$, since $\|\theta_{a_i}\|_\infty \to 0$. By (4.3), $v_i \to 0$ in L^∞. Using these facts and the equations for u_i and v_i, one can easily show by a compactness argument that

$$u_i/\|u_i\|_\infty \to \Phi_1, \quad v_i/\|v_i\|_\infty \to \Phi_1 \quad \text{in } C^1.$$

Thus we can rewrite u_i and v_i in the form

$$u_i = (s_i \cos\omega_i)(\Phi_1 + w_i), \quad v_i = (s_i \sin\omega_i)(\Phi_1 + z_i),$$

where

$$w_i,\ z_i \to 0 \text{ in } C^1,\ (w_i, \Phi_1)_2 = (z_i, \Phi_1)_2 = 0, \quad \omega_i \in (0, \pi/2),$$

and

$$s_i = \big(\|u_i\|_\infty^2/\|\Phi_1 + w_n\|_\infty^2 + \|v_i\|_\infty^2/\|\Phi_1 + z_i\|_\infty^2\big)^{1/2}.$$

We may assume that $\|h_i\|_2^2 + \|k_i\|_2^2 = 1$. Then, since $Re\eta_i \le 0$, as in the proof of Theorem 4.1, it can be easily shown that η_i is bounded. Thus by (4.1), h_i and

k_i are bounded in $W^{2,2}$. Without loss of generality, we may assume that $h_i \to h$, $k_i \to k$ in H_0^1 and $\eta_i \to \eta$. Passing to the weak limit in (4.1), we obtain

$$\begin{cases} \Delta h + \lambda_1 h + \eta h = 0, & h|_{\partial D} = 0, \\ \Delta k + \lambda_1 k + \tau\eta k = 0, & k|_{\partial D} = 0. \end{cases}$$

Thus we must have $\eta = 0$, $h = \zeta\Phi_1$ and $k = \xi\Phi_1$, where ζ and ξ are some real numbers. By Kato's inequality, we find that (4.2) still holds true. Passing to the limit in (4.2), we have

$$\int_D \Phi_1^2|k| \le c\int_D \Phi_1^2|h|.$$

Therefore $\zeta \ne 0$ and $|\xi| \le c|\zeta|$. Thus if we rescale (h_i, k_i) properly, we can assume that $h_i \to \Phi_1$, $k_i \to p\Phi_1$, where $|p| \le c$. Now dividing the equation for u_i by $s_i \cos\omega_i$, and multiplying it by Φ_1 and integrating by parts, we obtain

$$\lambda_1 \int_D \Phi_1^2 = a_i \int_D \Phi_1^2 - \int_D (\Phi_1 + w_i)\big(u_i + \frac{bv_i}{1+m_i u_i}\big)\Phi_1.$$

We may assume that $s_i m_i \to m^* \in [0,\infty]$ and $\omega_i \to \omega \in [0, \pi/2]$. Then

$$\lim_{i\to\infty} \frac{a_i - \lambda_1}{s_i}\int_D \Phi_1^2 = \cos\omega \int_D \Phi_1^3 + b\sin\omega \int_D \frac{\Phi_1^3}{1+m^*\cos\omega\Phi_1}. \tag{4.4}$$

Treating the equation for v_i analogously, we obtain

$$\lim_{i\to\infty} \frac{d_i - \lambda_1}{s_i}\int_D \Phi_1^2 = \sin\omega \int_D \Phi_1^3 - c\cos\omega \int_D \frac{\Phi_1^3}{1+m^*\cos\omega\Phi_1}. \tag{4.5}$$

Since $d_i \le \lambda_1$, it follows from (4.5) that $c\cos\omega \ge \sin\omega$. Moreover, if $\omega \ne 0$, then $m^* \ne \infty$. In particular, $\omega < \pi/2$. By rescaling (h_i, k_i) suitably once more if needed, we may assume that

$$h_i = \Phi_1 + h_i', \quad (\Phi_1, h_i')_2 = 0, \qquad h_i' \to 0.$$

Now multiplying the equation for h_i by Φ_1 and integrating over D, we obtain

$$\lambda_1 \int_D \Phi_1^2 = a_i \int_D \Phi_1^2 + \eta_i \int_D \Phi_1^2 - \int_D \big[2u_i + \frac{bv_i}{(1+m_i u_i)^2}\big] h_i\Phi_1 - \int_D \frac{bu_i k_i \Phi_1}{1+m_i u_i}.$$

From this we deduce that

$$\begin{aligned}\lim_{i\to\infty}\big(\frac{a_i-\lambda_1}{s_i} + \frac{\eta_i}{s_i}\big)\int_D \Phi_1^2 = 2\cos\omega \int_D \Phi_1^3 + b\sin\omega \int_D \frac{\Phi_1^3}{(1+m^*\cos\omega\Phi_1)^2} \\ + pb\cos\omega \int_D \frac{\Phi_1^3}{1+m^*\cos\omega\Phi_1}.\end{aligned}$$

Using this and (4.4), we easily obtain

$$\begin{aligned}\lim_{i\to\infty}\frac{\eta_i}{s_i}\int_D \Phi_1^2 = \cos\omega \int_D \Phi_1^3 + pb\cos\omega \int_D \frac{\Phi_1^3}{1+m^*\cos\omega\Phi_1} \\ - b\sin\omega \int_D \frac{m^*\cos\omega\Phi_1^4}{(1+m^*\cos\omega\Phi_1)^2}.\end{aligned} \tag{4.6}$$

Now we rewrite k_i in the form

$$k_i = p_i(\Phi_1 + k_i'), \; (\Phi_1, k_i')_2 = 0.$$

Then $p_i \to p$, and $k_i' \to 0$ if $p \neq 0$. We multiply the equation for k_i by Φ_1 and integrate by parts to obtain

$$p_i\lambda_1 \int_D \Phi_1^2 = p_i d_i \int_D \Phi_1^2 + \tau\eta_i p_i \int_D \Phi_1^2 - \int_D \big[2v_i - \frac{cu_i}{1+m_iu_i}\big] k_i\Phi_1 + \int_D \frac{cv_ih_i\Phi_1}{(1+m_iu_i)^2}.$$

Hence

$$\lim_{i\to\infty} p_i\big(\frac{d_i-\lambda_1}{s_i} + \tau\frac{\eta_i}{s_i}\big)\int_D \Phi_1^2 = 2p\sin\omega\int_D \Phi_1^3 - pc\cos\omega\int_D \frac{\Phi_1^3}{1+m^*\cos\omega\Phi_1} - c\sin\omega\int_D \frac{\Phi_1^3}{(1+m^*\cos\omega\Phi_1)^2}. \tag{4.7}$$

If $p_i \to p = 0$, then by (4.5) and (4.6), the left hand side of (4.7) is zero. Hence

$$c\sin\omega\int_D \Phi_1^3/(1+m^*\cos\omega\Phi_1)^2 = 0.$$

This implies that either $\omega = 0$ or $m^* = \infty$. In either case, (4.6) is reduced to

$$\lim_{i\to\infty}\frac{\eta_i}{s_i}\int_D \Phi_1^2 = \cos\omega\int_D \Phi_1^3 > 0,$$

contradicting $Re\eta_i \le 0$. Hence $p \neq 0$. Now we use

$$\int_D k_iv_i\big(d_i - v_i + \frac{cu_i}{1+m_iu_i}\big) = \int_D k_i(-\Delta v_i) = \int_D (-\Delta k_i)v_i$$
$$= \int_D k_iv_i(d_i - 2v_i + \frac{cu_i}{1+m_iu_i}) + \int_D \frac{cv_i}{(1+m_iu_i)^2}h_iv_i + \tau\eta_i\int_D k_iv_i,$$

to obtain

$$\int_D k_iv_i^2 = \int_D \frac{cv_i^2}{(1+m_iu_i)^2}h_i + \tau\eta_i\int_D k_iv_i.$$

Dividing the above identity by $s_i^2(\sin\omega_i)^2p$, taking the real parts and passing to the limit, we obtain

$$\int_D \Phi_1^3 \le Re(\frac{1}{p})c\int_D \frac{\Phi_1^3}{(1+m^*\cos\omega\Phi_1)^2}.$$

Hence $Re(p) > 0$. Thus by (4.6)

$$\cos\omega\int_D \Phi_1^3 - b\sin\omega\int_D \Phi_1^3\frac{m^*\cos\omega\Phi_1}{(1+m^*\cos\omega\Phi_1)^2} \le 0. \tag{4.8}$$

Since $\sin\omega \le c\cos\omega$ and

$$f(t) = \frac{t}{(1+t)^2} \le \frac{1}{4}$$

for all $t \ge 0$, with equality holding if and only if $t = 1$, we have

$$\frac{m^*\cos\omega\Phi_1}{(1+m^*\cos\omega\Phi_1)^2} \le \frac{1}{4},$$

and by (4.8),

$$\cos\omega\int_D \Phi_1^3 - (1/4)bc\cos\omega\int_D \Phi_1^3 < 0.$$

This gives $bc > 4$, a contradiction. The proof is complete. #

It is natural to ask whether $bc \leq 4$ is only a technical condition in Theorem 4.2. It turns out that this is not the case. To be more precise, if m is bounded, then this condition is not needed, but for large m, a restriction of the type $bc < B$ for some $B > 0$ is necessary. We first consider the case when m is bounded, and in fact, we establish a more general result.

Theorem 4.3. *For any given $M > 0$, there exists ϵ small, depending only on b, c and M, such that for any $a \in (\lambda_1, \lambda_1 + \epsilon)$ and $m \in [0, M]$, (1.1) has no positive solution for $d \notin (d_0, d_1)$, and has a unique positive solution for $d \in (d_0, d_1)$. Moreover, the unique positive solution is asymptotically stable. Here d_0 and d_1 are defined as in Lemma 2.6.*

Proof. We first show that for any given $M > 0$, there exists $\epsilon > 0$ small such that for $a \in (\lambda_1, \lambda_1 + \epsilon)$, $m \in [0, M]$ and any d, any positive solution of (1.1) is non-degenerate and linearly stable.

This is just a modification of the proof of Theorem 4.2, and in fact it is much simpler. Note that here we do not have the restriction $d \leq \lambda_1$.

We again argue indirectly. Suppose that for some given $M > 0$, we can find $a_i \to \lambda_1 +$, $m_i \in [0, M]$ and d_i such that (1.1) with $(a, d, m) = (a_i, d_i, m_i)$ has a positive solution (u_i, v_i) which is either degenerate or unstable. That is, there exist η_i with $Re\eta_i \leq 0$, (h_i, k_i) smooth and $(h_i, k_i) \neq (0, 0)$ such that (4.1) holds.

Since $0 \leq u_i \leq \theta_{a_i}$, we have $u_i \to 0$ in L^∞. From the equations for u_i and v_i, we again deduce that (4.3) remains true. Hence it follows from $v_i \neq 0$ that

$$d_i > \lambda_1 - c\|\theta_{a_i}\|_\infty \to \lambda_1.$$

We may assume by choosing a subsequence that $d_i \to d^* \in [\lambda_1, \infty]$. Then by (4.3), $v_i \to \theta_{d^*}$ in L^∞. Hence

$$u_i + bv_i/(1 + m_i u_i) \to b\theta_{d^*}.$$

Here we understand that $\theta_\infty = \infty$. But it follows from $u_i > 0$ and the equation for u_i that

$$\lambda_1\big(u_i + bv_i/(1 + m_i u_i)\big) = a_i.$$

Passing to the limit, we obtain $\lambda_1(b\theta_{d^*}) = \lambda_1$. Hence we must have $d^* = \lambda_1$, which implies that $d_i \to \lambda_1$. By (4.3), this in turn gives $v_i \to 0$ in L^∞.

Now we see clearly that everything in the proof of Theorem 4.2 carries over to the present case except those following from the assumption $d_i \leq \lambda_1$. Also note that now we have $m^* = 0$. In particular, $Re(p) > 0$ remains true. But on the other hand,

$$\begin{aligned}\int_D h_i u_i\big(a_i - u_i - \frac{bv_i}{1 + m_i u_i}\big) &= \int_D (-\Delta u_i) h_i = \int_D (-\Delta h_i) u_i \\ &= \int_D h_i u_i\big[a_i - 2u_i - \frac{bv_i}{(1 + m_i u_i)^2}\big] - \int_D \frac{bu_i}{1 + m_i u_i} k_i u_i + \eta_i \int_D u_i h_i.\end{aligned}$$

Hence

$$\int_D u_i^2 h_i = \int_D bv_i \frac{m_i u_i}{(1 + m_i u_i)^2} h_i u_i - \int_D \frac{bu_i}{1 + m_i u_i} k_i u_i + \eta_i \int_D u_i h_i.$$

Dividing the above identity by $s_i^2(\cos\omega_i)^2$, and using the expression for u_i, we have

$$\int_D (\Phi_1 + w_i)^2 h_i = \int \frac{bv_i m_i}{(1+m_i u_i)^2}(\Phi_1 + w_i)^2 h_i - \int_D \frac{b}{1+m_i u_i}(\Phi_1 + w_i)^2 k_i + \frac{\eta_i}{s_i \cos\omega_i}\int_D (\Phi_1 + w_i) h_i.$$

Then taking the real parts and passing to the limit in the above identity, we deduce that

$$\int_D \Phi_1^3 \leq -bRe(p)\int_D \Phi_1^3.$$

Thus $Re(p) < 0$. This contradiction completes the proof of our claim.

By Lemma 2.6, our claim implies that for $d \in (d_0, d_1)$, (1.1) has a unique positive solution and it is asymptotically stable.

It remains to show that there is no positive solution if $d \notin (d_0, d_1)$. Recall that if a_0 is defined as in Proposition 4.1, then $d \leq d_0$ if and only if $a \leq a_0$. Hence it follows from Proposition 4.1 that (1.1) has no positive solution if $d \leq d_0$.

Next we show that (1.1) has no positive solution if $d \geq d_1$. We argue indirectly. Suppose that for some $a' \in (\lambda_1, \lambda_1 + \epsilon)$ and $m' \in [0, M]$, there exists $d' \geq d_1$ so that (1.1) has a positive solution for $(a, m, d) = (a', m', d')$. Then set

$$d^* = \sup\big\{d'' :\ (1.1) \text{ has a positive solution for } (a, m, d) = (a', m', d'')\big\}.$$

Clearly $d^* \geq d' \geq d_1$. By Theorem 4.2 in [2], we have $d^* < \infty$. There are only two possibilities: (a) $d^* = d_1$ and (b) $d^* > d_1$.

In case (a), we must have $d^* = d' = d_1$. Thus (1.1) has a positive solution (u^*, v^*) for $d = d^*$. By our claim as shown in the above, (u^*, v^*) is non-degenerate. Hence it follows from the implicit function theorem that (1.1) has a unique solution (u, v) near (u^*, v^*) in the C^1 norm if d is close to d^*. It then follows from the maximum principle that (u, v) is a positive solution. But this contradicts the definition of d^*.

In case (b), it follows from a simple compactness argument that (1.1) has a non-negative solution (u^*, v^*) at $d = d^*$. If (u^*, v^*) is a positive solution, then we arrive at a contradiction in the same way as in case (a). Therefore we may suppose that either $u^* = 0$ or $v^* = 0$. Then there must exist $d_i \to d^*$ and positive solutions (u_i, v_i) of (1.1) with $d = d_i$ such that $(u_i, v_i) \to (u^*, v^*)$ in the C^1 norm.

If $u^* = 0$, then we easily deduce that $v_i \to \theta_{d^*}$. Hence

$$a = a' = \lambda_1\big(u_i + bv_i/(1 + m_i u_i)\big) \to \lambda_1(b\theta_{d^*}),$$

which is impossible since

$$\lambda_1(b\theta_{d^*}) > \lambda_1(b\theta_{d_1}) = a.$$

If $v^* = 0$, then $u_i \to \theta_a$. Therefore

$$d_i = \lambda_1\big(v_i - cu_i/(1 + m_i u_i)\big) \to \lambda_1\big(-c\theta_a/(1 + m\theta_a)\big).$$

But $d_i \to d^*$. Thus

$$d^* = \lambda_1(-c\theta_a/(1 + m\theta_a)) = d_0 < d^*.$$

Again we have a contradiction. This completes the proof of Theorem 4.3. #

Remark 4.2. 1) Theorem 4.3 improves a result of Yamada [28], where only the existence of a stable solution is proved and is for the case when $m = 0$.

2) Theorem 4.3 contrasts with the case when m is large. In the large m case, one can show by using ideas of this paper that, given any $a > \lambda_1$ and any $D > d_1$, there exists m large such that (1.1) has at least two positive solutions for $d \in (d_1, D]$.

To conclude this section, we show that Theorem 4.2 is no longer true if bc is large. More precisely, we prove the following result.

Theorem 4.4. *There exists $M_0 > 0$ large such that for any $b > 0, c > 0$ satisfying $bc \geq M_0$, we can find $\zeta > 0, \eta \geq 0$ and $\epsilon_0 > 0$ such that for any $m > 1/\epsilon_0, a = \lambda_1 + \zeta/m$ and $d = \lambda_1 - \eta/m$, (1.1) has at least three non-degenerate positive solutions.*

To this end, we first make a change of variables in (1.1). Let $\tilde{u} = u/\epsilon, \tilde{v} = v/\epsilon, a = \lambda_1 + \zeta\epsilon, d = \lambda_1 - \eta\epsilon, m = 1/\epsilon$, where $\epsilon > 0, \zeta > 0, \eta \geq 0$. Then (1.1) is equivalent to

$$\begin{cases} -\Delta\tilde{u} = \lambda_1\tilde{u} + \epsilon\big(\zeta\tilde{u} - \tilde{u}^2 - \dfrac{b\tilde{u}\tilde{v}}{1+\tilde{u}}\big), & \tilde{u}|_{\partial D} = 0, \\ -\Delta\tilde{v} = \lambda_1\tilde{v} + \epsilon\big(-\eta\tilde{v} - \tilde{v}^2 + \dfrac{c\tilde{u}\tilde{v}}{1+\tilde{u}}\big), & \tilde{v}|_{\partial D} = 0. \end{cases} \tag{4.9}$$

For ϵ small and ζ, η fixed, (4.9) is a smooth perturbation of the problem

$$\begin{cases} -\Delta\tilde{u} = \lambda_1\tilde{u}, & \tilde{u}|_{\partial D} = 0, \\ -\Delta\tilde{v} = \lambda_1\tilde{v}, & \tilde{v}|_{\partial D} = 0, \end{cases}$$

for which positive solutions form a two dimensional manifold

$$M = \big\{(t\phi_1, s\phi_1) : t, s > 0\big\},$$

where $\phi_1 = \alpha\Phi_1$ and $\alpha > 0$ is chosen so that $\int_D \phi_1^3 = 1$. The unusual normalization here is chosen just for the convenience of later calculations.

We use a technique of Dancer [12] to study the positive solutions of (4.9) with $\epsilon > 0$ small. The main idea is that for ϵ small, positive solutions of (4.9) near M can be determined by the zeros of a mapping on M obtained by the perturbation term in (4.9). In general, the zeros of the mapping on M are easy to analyze. To be more precise, we will make use of the following result from [12], page 430.

Proposition 4.2. *Suppose that H is a smooth Fredholm map between Banach spaces X and Y, and H vanishes on a smooth manifold M with*

$$dim N\big(H'(x)\big) = dim M, \quad x \in M.$$

Let P_x be a smooth projection of Y onto $R\big(H'(x)\big)$ for $x \in M$. Suppose further that $J : X \times R \to Y$ is smooth. Then the zeros of

$$R(x, \epsilon) = H(x) + \epsilon J(x, \epsilon)$$

near M for ϵ small are determined by the zeros of the map

$$G(x) = (I - P_x)J(x, 0)$$

on M in the sense that, for any zero (x, ϵ) of R with $\epsilon \neq 0$ small, x must be close to a zero of G; conversely, if H has index 0, then near any non-degenerate zero of G, we have a unique zero of R for each small non-zero ϵ, and this zero of R is non-degenerate.

Proof of Theorem 4.4. Set $X = C_0^{2,\alpha}(\overline{D}) \times C_0^{2,\alpha}(\overline{D})$ and $Y = C^\alpha(\overline{D}) \times C^\alpha(\overline{D})$ for some $\alpha \in (0,1)$. Define operators $H : X \to Y$, $Q : Y \to M$ and $J : X \to Y$ by

$$H(u,v) = (\Delta u + \lambda_1 u, \Delta v + \lambda_1 v),$$
$$Q(u,v) = \Big(\frac{\int_D u\phi_1}{\int_D \phi_1^2}\phi_1,\ \frac{\int_D v\phi_1}{\int_D \phi_1^2}\phi_1\Big),$$
$$J(u,v) = \Big(\zeta u - u^2 - \frac{buv}{1+u}, -\eta v - v^2 + \frac{cuv}{1+u}\Big)$$

and set $P_x = I - Q$. Then it is easy to check that (4.9) is equivalent to

$$R(u,v,\epsilon) \equiv H(u,v) + \epsilon J(u,v) = 0$$

and that all the conditions of Proposition 4.2 are satisfied. Now we have $G(u,v) = QJ(u,v)$ and, for $(u,v) = (t\phi_1, s\phi_1) \in M$,

$$G(u,v) = G(t\phi_1, s\phi_1) = \frac{\phi_1}{\int_D \phi_1^2}\big(tf(t,s), sg(t,s)\big),$$

where

$$f(t,s) = \zeta \int_D \phi_1^2 - t - bs\int_D \frac{\phi_1^3}{1+t\phi_1},$$
$$g(t,s) = -\eta \int_D \phi_1^2 - s + ct\int_D \frac{\phi_1^3}{1+t\phi_1}.$$

Clearly $(t\phi_1, s\phi_1)$ with $t \neq 0$ and $s \neq 0$ is a zero of G if and only if

$$f(t,s) = g(t,s) = 0,\ t \neq 0, s \neq 0.$$

If we define $F : R^2 \to R^2$ by $F(t,s) = \big(tf(t,s), sg(t,s)\big)$, then by Proposition 4.2, any non-degenerate zero of (t_0, s_0) of F with $t_0 > 0, s_0 > 0$ gives a solution of (4.9) near $(t_0\phi_1, s_0\phi_1)$ (hence it is a positive solution) for all small ϵ. Therefore in order to prove what we want, it suffices to show that for certain large M_0, if $bc \geq M_0$, then we can find $\zeta > 0$ and $\eta \leq 0$ such that F has at least three non-degenerate zeros with positive components.

Though not necessary, for the simplicity of calculations, we choose $\eta = 0$. Then $F(t,s) = 0$ with $t \neq 0, s \neq 0$ is equivalent to

$$\zeta \int_D \phi_1^2 - t - bct\Big(\int_D \frac{\phi_1^3}{1+t\phi_1}\Big)^2 = 0,$$
$$s - ct\int_D \frac{\phi_1^3}{1+t\phi_1} = 0.$$

Set

$$h(t) = \zeta \int_D \phi_1^2 - t - bct\Big(\int_D \frac{\phi_1^3}{1+t\phi_1}\Big)^2.$$

Clearly $h(0) > 0$ and $\lim_{t\to\infty} h(t) = -\infty$. This implies that $h(t) = 0$ has at least one positive root for $\zeta > 0$ and $bc > 0$. We show that if we choose bc large and ζ properly, then h has at least three positive roots: t_1, t_2 and t_3. Then for

$$s_i = ct_i\int_D \frac{\phi_1^3}{1+t_i\phi_1},$$

(t_i, s_i) are three zeros of F with positive components. If we can also show that (t_i, s_i) are non-degenerate zeros of F, then by our previous discussions, we are done.

A simple calculation gives

$$h^{'}(t) = -1 + bc\int_D \frac{\phi_1^3}{1+t\phi_1}\int_D \frac{(t\phi_1 - 1)\phi_1^3}{(1+t\phi_1)^2}.$$

It is not hard to see that there exists $t_0 > 0$ such that

$$\int_D \frac{(t\phi_1 - 1)\phi_1^3}{(1+t\phi_1)^2} > 0 \text{ for all } \ t \geq t_0.$$

Now we may choose $M_0 > 0$ large so that $h^{'}(t_0) > 0$ for all $bc \geq M_0$. Then we have

$$h^{'}(0) < 0,\ h^{'}(t_0) > 0,\ \lim_{t\to\infty} h^{'}(t) = -1.$$

Hence by elementary calculus, if we choose $\zeta > 0$ properly, $h(t) = 0$ has at least three roots $0 < t_1 < t_2 < t_3 < \infty$. It follows from Sard's theorem that, by perturbing ζ when necessary, we can make sure that the t_i are non-degenerate zeros of h, i.e., $h^{'}(t_i) \neq 0$ for $i = 1, 2, 3$. It remains to show that (t_i, s_i) are non-degenerate zeros of F. In fact, by simple calculations, the Jacobian

$$\frac{\partial F(t_i, s_i)}{\partial (t,s)} = -t_i s_i h^{'}(t_i) \neq 0 \quad \text{ for } i = 1, 2, 3.$$

Thus (t_i, s_i) are indeed non-degenerate. This completes the proof.#

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